2D Fourier, Scale, and Cross-correlation

CS 510

Lecture #12

February 26th, 2014

Colorado State University

Where are we?

- We can detect objects, but they can only differ in translation and 2D rotation
- Then we introduced Fourier analysis.
- Why?
  - Because Fourier analysis can help us with scale
  - Because Fourier analysis can make correlation faster

Review: Discrete Fourier Transform

- Problem: an image is not an analogue signal that we can integrate.
- Therefore for 0 ≤ x < N and 0 ≤ u <N/2:

$$F(u) = \sum_{x=0}^{N-1} f(x)\left[\cos\left(\frac{2\pi ux}{N}\right) - i\sin\left(\frac{2\pi ux}{N}\right)\right]$$

And the discrete inverse transform is:

$$f(x) = \frac{1}{N}\sum_{x=0}^{N-1} F(u)\left[\cos\left(\frac{2\pi ux}{N}\right) + i\sin\left(\frac{2\pi ux}{N}\right)\right]$$

2D Fourier Transform

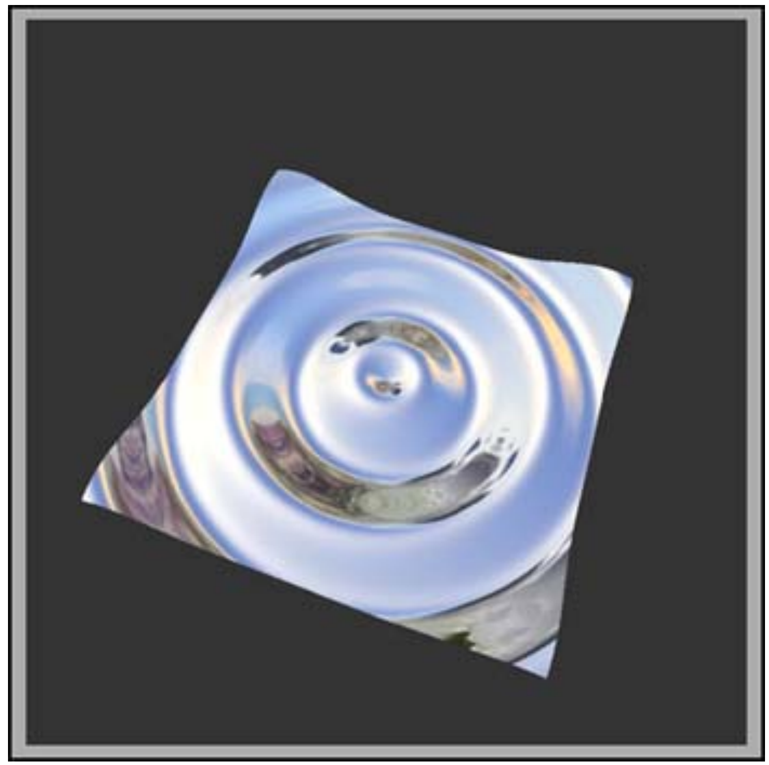
- So far, we have looked only at 1D signals
- For 2D signals, the continuous generalization is:

$$F(u,v) \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\left[\cos\left(2\pi(ux+vy)\right) - i\sin\left(2\pi(ux+vy)\right)\right]$$

- Note that frequencies are now two-dimensional
  - u= freq in x, v = freq in y
- Every frequency (u,v) has a real and an imaginary component.

2D sine waves

- This looks like you'd expect in 2D
  - ➢ Note that the frequencies don't have to be equal in the two dimensions.



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2D Discrete Fourier Transform

$$F(u,v) = \sum_{x=-N/2}^{N/2} \sum_{y=-N/2}^{N/2} f(x,y)\left[\cos\left(\frac{2\pi}{N}(ux+vy)\right) - i\sin\left(\frac{2\pi}{N}(ux+vy)\right)\right]$$

- What happened to the bounds on x & y?
- How big is the discrete 2D frequency space representation?

2D Frequency Space

- Remember that:
  - Cosine is an even function: $\cos(x) = \cos(-x)$
  - Sine is an odd function: $\sin(x) = -\sin(-x)$
- So
  - $F(u,v) = a+ib \Rightarrow F(-u, -v) = a-ib$
- And
  - $F(-u,v) = a+ib \Rightarrow F(u, -v) = a-ib$
- But
  - $F(u,v) = a+ib \Rightarrow F(-u, v) = ???$

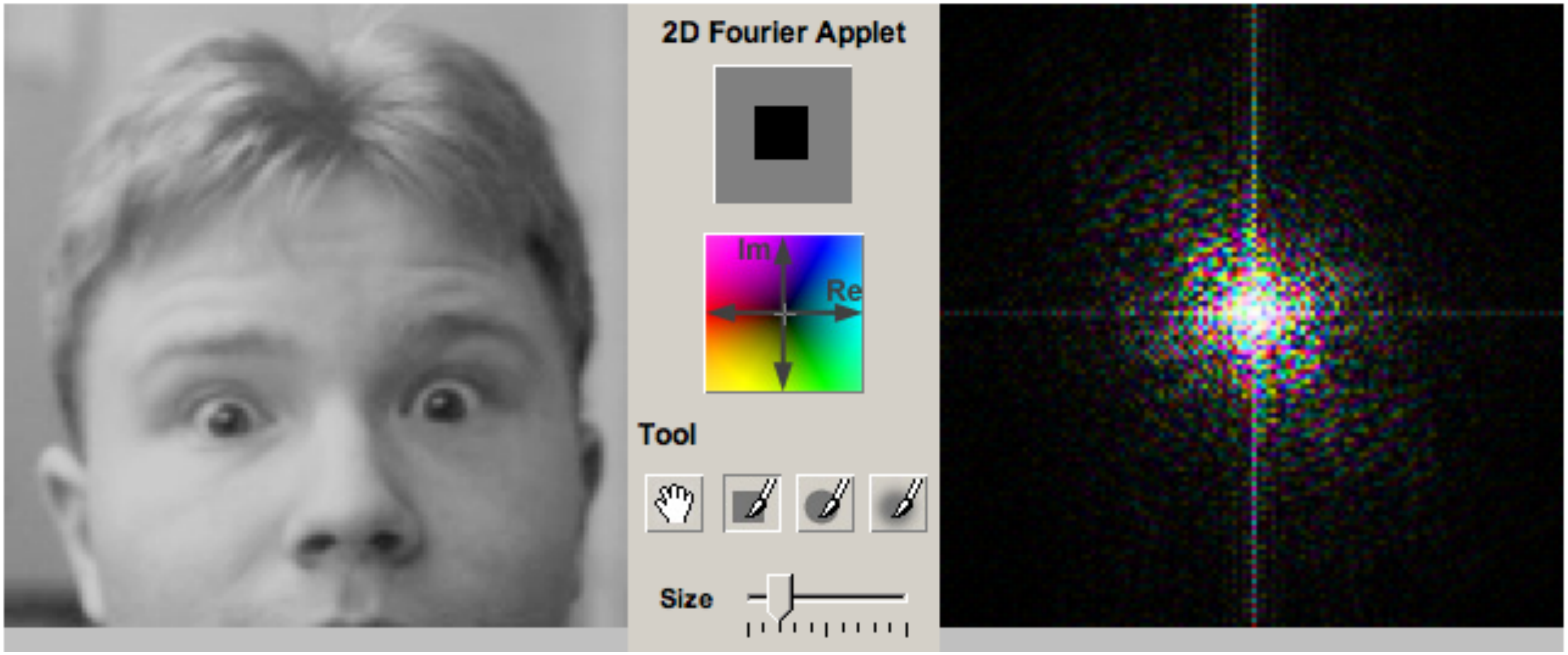
2D Frequency Space (cont)

- Size of 2D Frequency representation:
  - One dimension must vary from –N/2 to N/2, while the other varies from 0 to N/2
    - Doesn't matter which is which
  - N * (N/2) * 2 values per frequency = N^2
  - Same as the source spatial representation

Showing Frequency Space

- To display a frequency space:
  - We plot it from –N/2 to N/2 in both dimensions
  - The result is symmetric about the origin (and therefore redundant)
  - We can't plot a complex number, so we show the magnitude at every pixel sqrt(a^2 + b^2)
    - Thus discarding the phase information
    - Phase plots are also possible ($\tan^{-1}(b/a)$)

Showing Frequency Space



http://www.brainflux.org/java/classes/FFT2DApplet.html

But Why?

- Reason 1: Fast Correlation

- Reason 2: Scale

Review: Convolution

We arrive at the fundamental idea of convolution.

“Slide” a mask over an image. At each window position, multiply the mask values by the image value under them.

Sum the results for every pixel.

Think of this as a sliding dot product

Convolution (cont.)

- Why return to convolution after introducing the Fourier Transform?

- Because multiplying two signals in the frequency domain is the same as convolving them in the spatial domain! (trust me)

Computing Cross-Correlation

- In cross-correlation, the mask is convolved with the target image
  - zero-mean & unit length the mask
  - zero-mean & unit length the image
  - Convolve the image and mask

Fast correlation

- If we compute correlation in the spatial domain, the cost is O(nm), where n > m.
- What if we use the frequency domain?
  - Convolution becomes point-wise multiplication
  - Convert to frequencies: O(n log n)
  - Point-wise multiply: O(n)
  - Convert back to spatial: O(n log n)
- Frequency domain is faster if log(n) < m

Fast correlation (II)

- Is spatial convolution really the same as frequency point-wise multiplication?
- Yes, but…
  - Take the complex conjugate of the mask
  - Images must be the same size
    - Pad mask with zeroes
    - Doesn't change the overall complexity
  - What happens at the image edges?
    - Frequency domain repeats
    - Values off the source image aren't zero
    - Equivalent to convolution on a torus

Fourier Correlation

- Simple convolution, not Pearson's correlation
  - The template can be zero mean & unit length
  - But the image windows won't be
- No 2D rotation
- But fast! O(n log(n))

Using Fourier Correlation

- Generate multiple templates at different rotations
- Pad to image size
- Multiply with target in frequency domain
- Find peak in spatial domain
  – Not true correlation
  – Only rough rotation
  – But fast
- Perform true rotation & correlation at peaks

But Why?

- Reason 1: Fast Correlation

- Reason 2: Scale

Reminder…

$$
\begin{aligned}
g(x) = {} & a_1 \cos(f_1 x) + b_1 \sin(f_1 x) \\
& + a_2 \cos(f_2 x) + b_2 \sin(f_2 x) \\
& + a_3 \cos(f_3 x) + b_3 \sin(f_3 x) \\
& + \cdots
\end{aligned}
$$

- Signal is reconstructed as a series of sine and cosine waves

Review: Fourier Magnitude & Phase

- The energy at a frequency is:

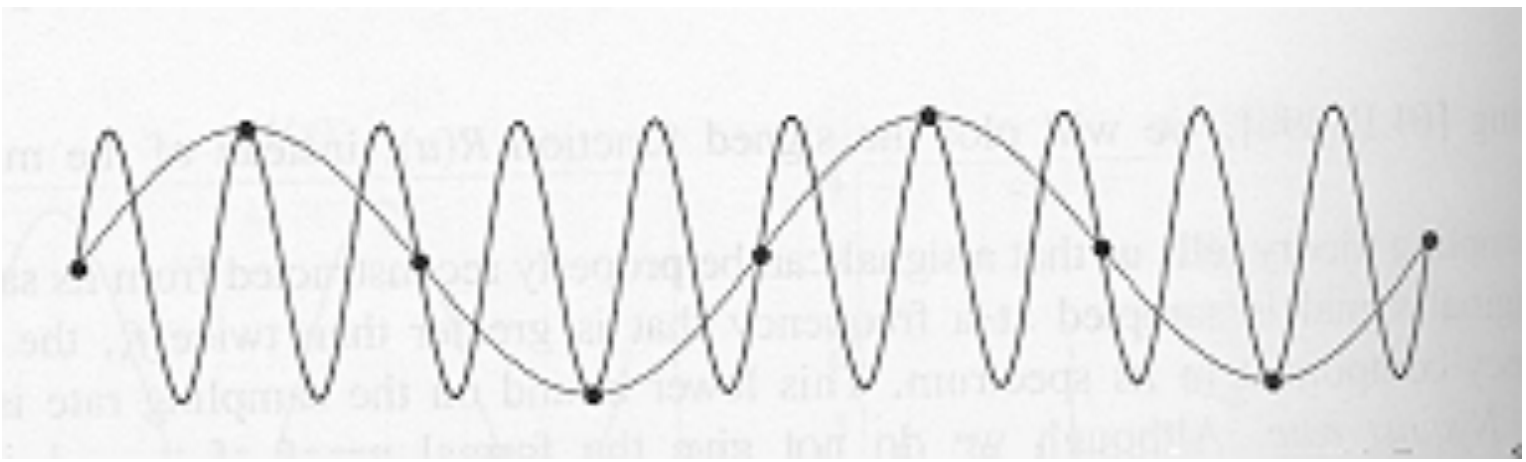
$$|F(u)| = \sqrt{R^2(u) + I^2(u)}$$

- The phase at a frequency is:

$$\tan^{-1}(u) = \frac{I(u)}{R(u)}$$

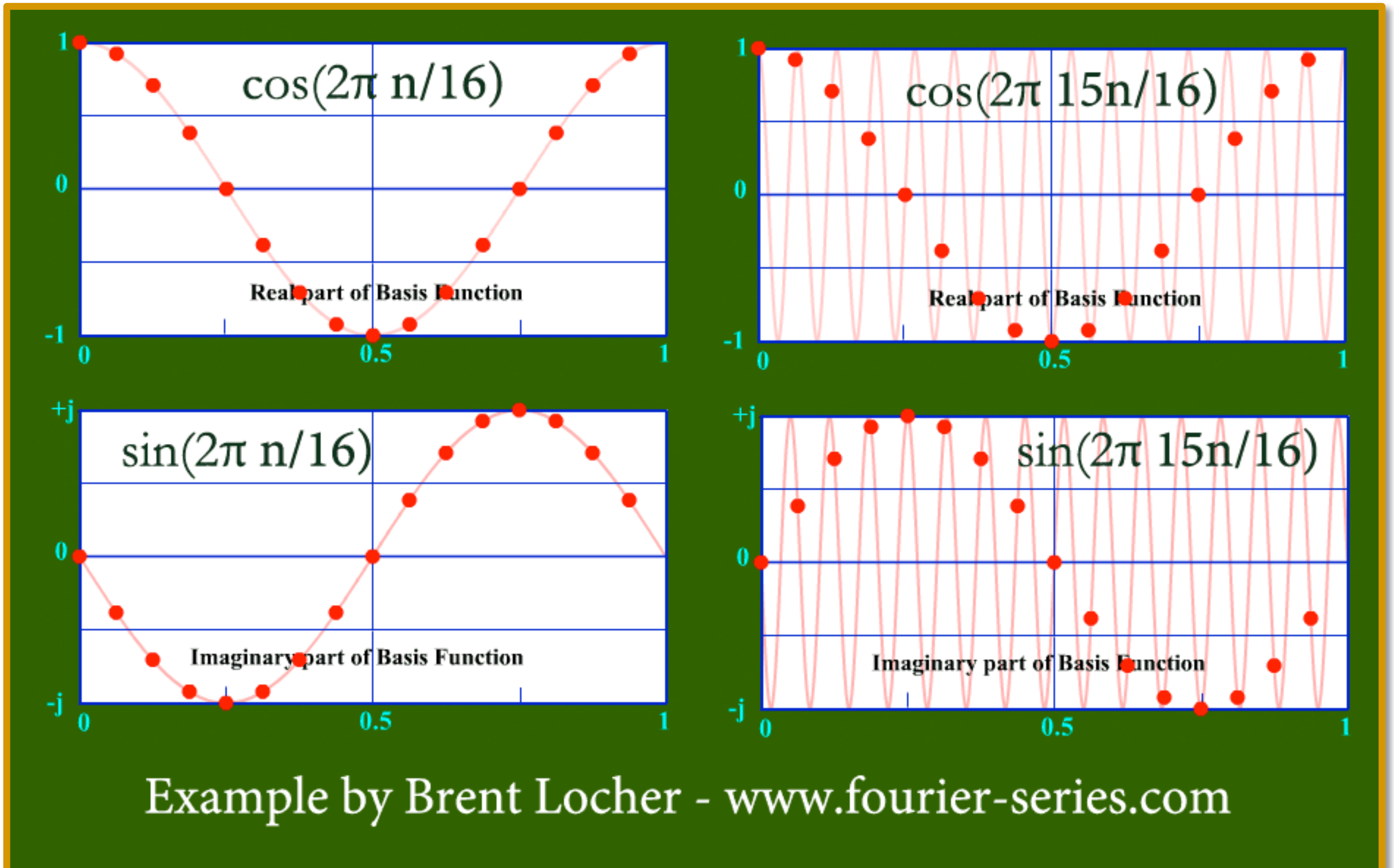
The Nyquist Rate

- What if the frequency is above N/2?
  - You have fewer than one sample per half-cycle
  - High frequencies look like lower frequencies



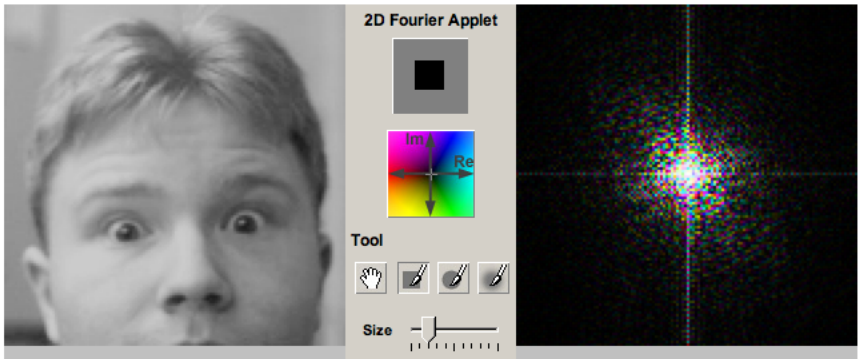
Graphic from *"Computer Graphics:Principles and Practice" by Foley, van Dam, Feiner & Hughes.*

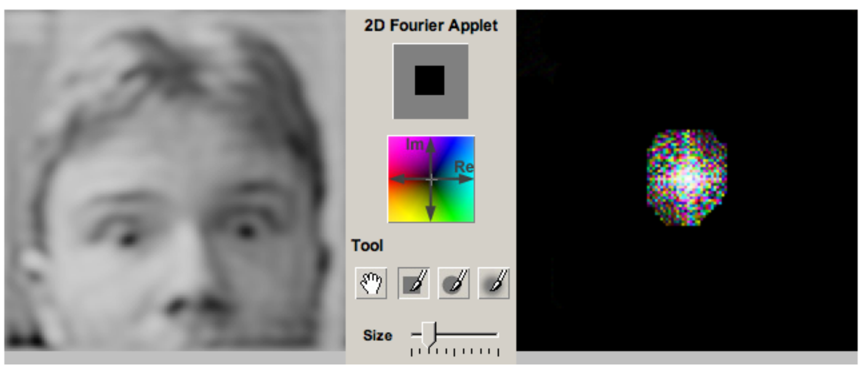
Aliasing – Another View



Low-Pass Filtering 101

- Drop high frequency Fourier coefficients.





To low-pass filter an image:
1) convert to frequency domain
2) discard all values for u > thresh
3) Convert back to spatial domain

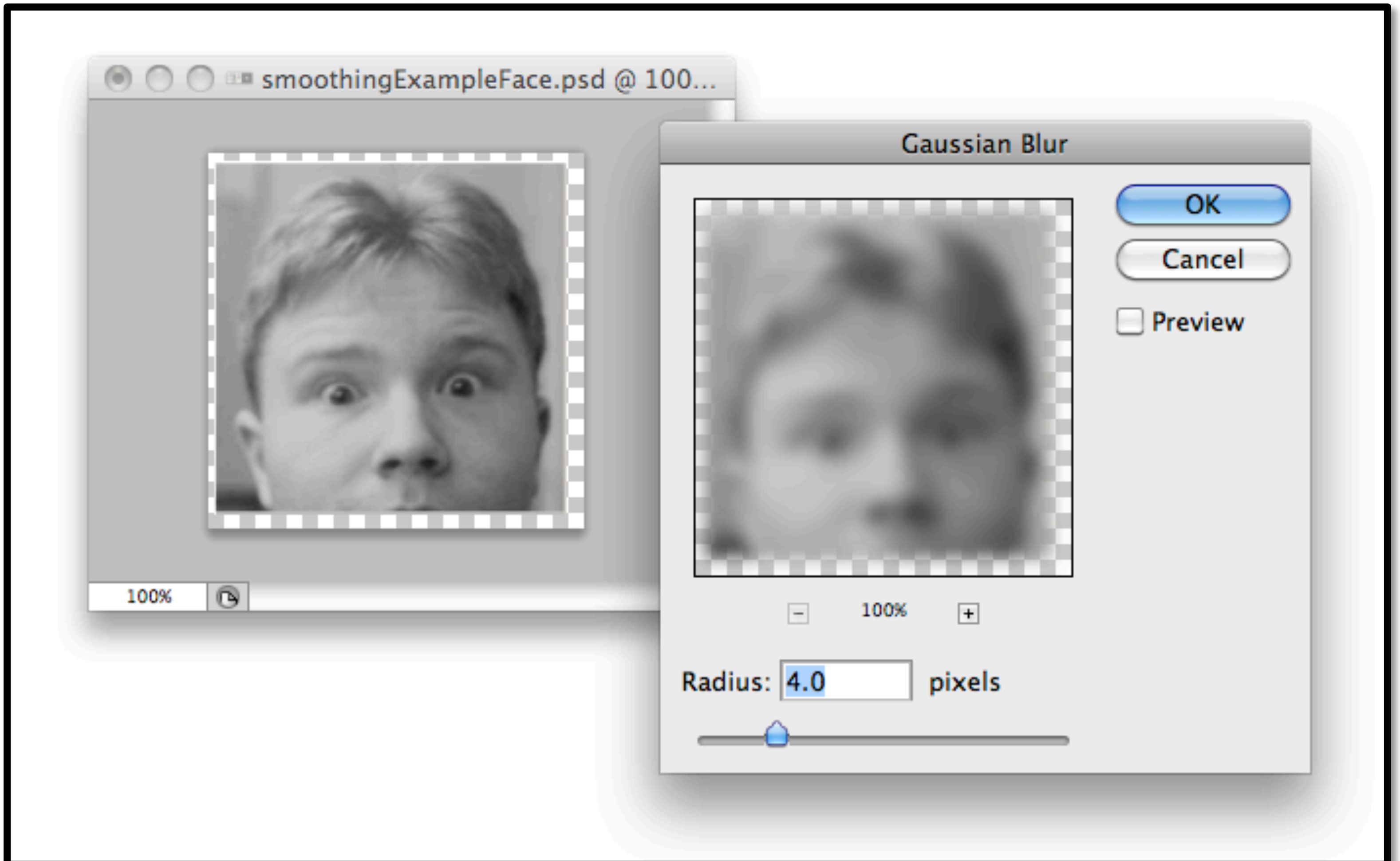
Brainflux Fourier Applet

http://www.brainflux.org/java/classes/FFT2DApplet.html

But is there an easier way?
A more efficient way?

➢ Alternatively, convolve with a Gaussian.

Photoshop Gaussian Blur

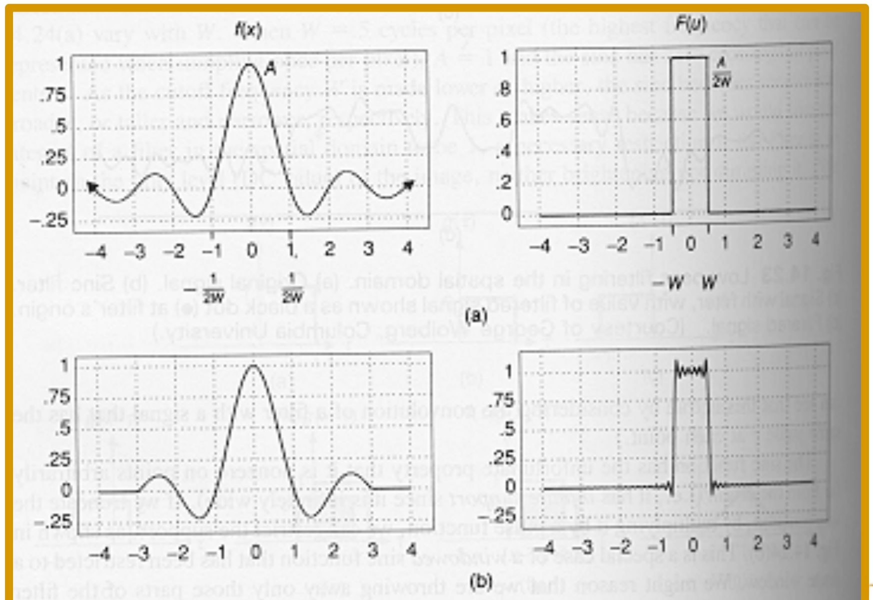


Low-Pass Filter

- Low-Pass filter - multiply by a pulse in frequency space, or
- Convolve the image with the inverse Fourier transform of a pulse...

Sinc filter →

Truncated sinc →

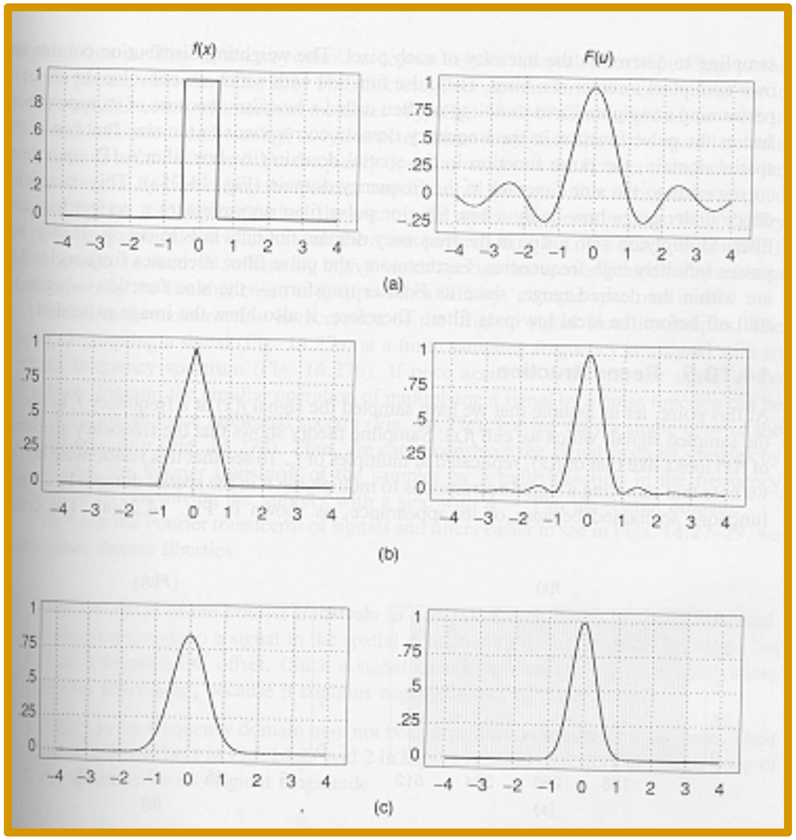


Graphic from *"Computer Graphics:Principles and Practice" by Foley, van Dam, Feiner & Hughes.*

The Gibbs Phenomenon (ringing)

- The truncated sinc is no longer a pulse in frequency space
  - passes small amounts of some high frequencies
  - passes acceptable frequencies in uneven amounts
  - may create negative values in unusual circumstances

Alternative Filters



pulse/sinc

triangle/sinc^2

gaussian/gaussian

Graphic from *"Computer Graphics:Principles and Practice" by Foley, van Dam, Feiner & Hughes.*

Image Reductions

- Anytime the target image has a lower resolution than the source image, prevent frequency aliasing by low-pass filtering.
  - In practice, convolve with a Gaussian
  - Determine Nyquist rate for target image
    - ½ width and ½ height
  - Select σ
  - Convolve source image with $g(\sigma)$
  - Apply geometric transformation to result

Image Reductions (II)

- Example: reduce 1Kx1K to 800x800 pixels
  - Select one (source) pixel as unit length
  - The Nyquist rate for source is 0.5 cycles/s_pixel
  - Nyquist rate for target is 0.4 cycles/s_pixel
- Problem: Gaussian is not a strict cut-off
  - Select "pass" value (2σ sounds good)
  - Select mask width to cover "most" of the area under the Gaussian curve
    - recommend 5σ (source: Trucco & Verri)
    - Covers 98.75% of the area under the Gaussian

Image Reduction (III)

- So 2σ is 0.4 cycles/pixel
  - The Fourier transform of $g(x, \sigma)$ is $g(\omega, 1/\sigma)$
  - The inverse of 0.4 cycles/pixel is 2.5 pixels/cycle
    - 2σ = 2.5 pixels/cycle
    - σ = 1.25 pixels/cycle
  - (T&V): To include 5σ of the curve, $\sigma = w/5$,
    - w is the width of the mask
    - W = 6.25
- Create a 7x7 Gaussian mask with sigma 1.25
  - w should be odd, so don't use 6x6
    - Why make w odd? To avoid a geometric transformation…
- Smooth the image using this mask, then subsample.

Image Transformation

- What if we want to keep 1Kx1K size?
  - Target Nyquist rate is 0.5 cycles/pixel
  - In image space, 2σ = 2 pixels/cycle, so $\sigma=1$
  - $\sigma = w/5$, so $w = 5$
  - Create a 5x5 mask with $\sigma=1$, smooth source image
  - Transform (rotate, etc.) the result.
- This is why most image processing packages includes predefined 5x5 Gaussian masks
- Other masks you build yourself.

Smoothing with σ=1

Original Image

Image with Gaussian Smoothing, σ = 1.0

Limits to Gaussians

- The Gaussian mask itself is a discrete sampling of a continuous signal.
- Gaussian signals with sigmas below 0.8 are too small to be sampled at pixel intervals.
- Generally not used for “up-sampling”

Implications of Smoothing

- All of this is based on the view that an image is a sum of sine waves.
- Physically, this assumption is absurd
  - Think of a ray tracer -- where would sine waves (or repeating signals) come from?
  - Occlusion edges lead to non-differentiable jumps
    - the signal content on the two sides are unrelated
    - violates the differentiability assumption underlying Fourier analysis
  - Edges are therefore very high frequency;
    - G(x, σ=1) blurs the image
- Fourier analysis does describe the limitations of A/D conversion, and therefore of image manipulation