The Structure Tensor (or, a gentle intro to PCA)

Lecture #09

February 12th, 2014

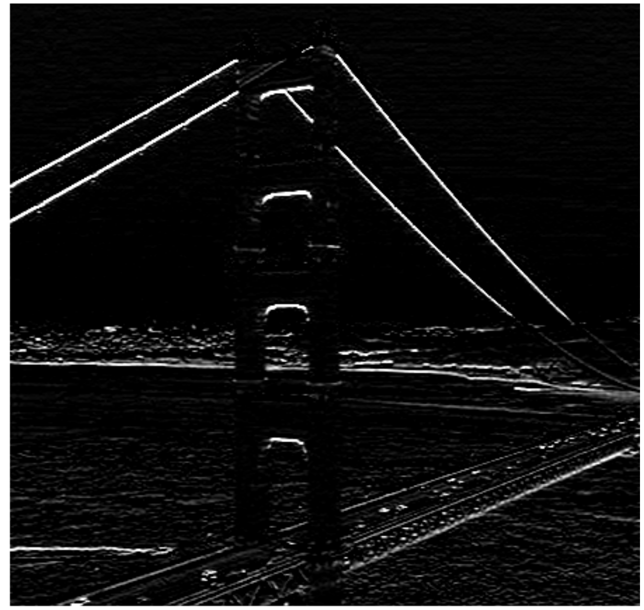
Colorado State University

Review: Edges

- Convolution with an edge mask estimates the partial derivatives of the image surface.
- The Sobel edge masks are:

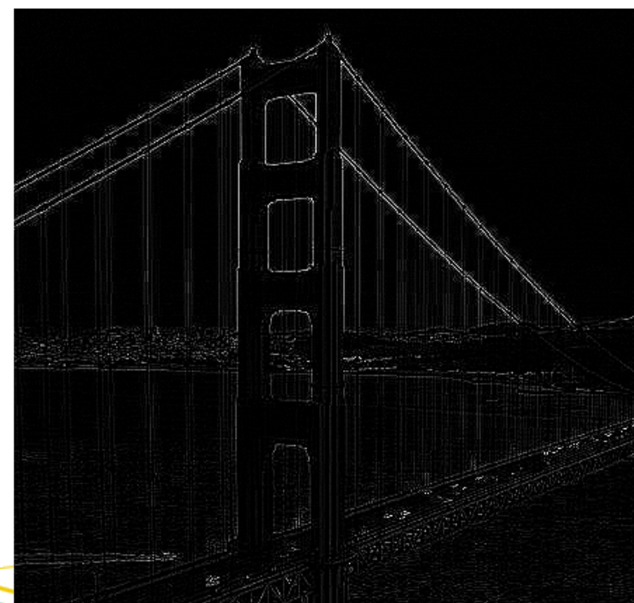
$$\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Dx Dy



$$\begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$



$$\begin{vmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{vmatrix}$$

Using Dx & Dy (Review)

- Convolution produces two images
  - One of partial derivatives in dI/dx
  - One of partial derivatives in dI/dy
- At any pixel (x,y):

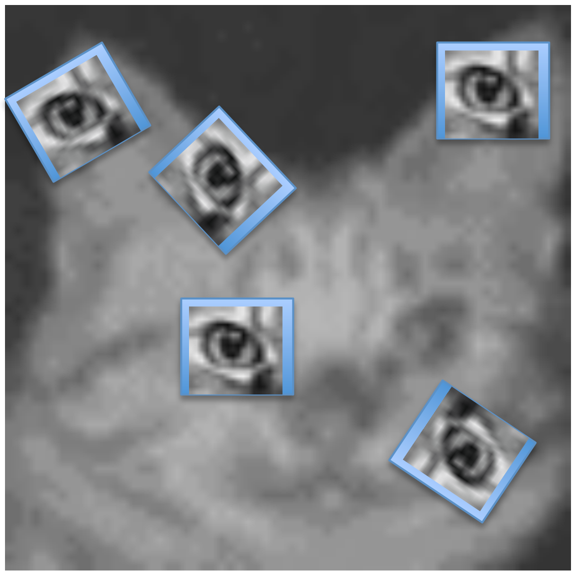
$$EdgeMagnitude(x,y) = \sqrt{\left(\partial I/\partial x\right)^2 + \left(\partial I/\partial y\right)^2}$$

$$EdgeOrientation(x,y) = \tan^{-1}\left(\frac{\partial I/\partial y}{\partial I/\partial x} \right)$$

Rotation-Free Correlation

- Pre-process: center the template on an edge
- For every Image window:
  - Measure the direction of the edge at the center pixel
  - Rotate the template until its center pixel has the same orientation
  - Correlate the template & image window

Rotation-Free Correlation (II)

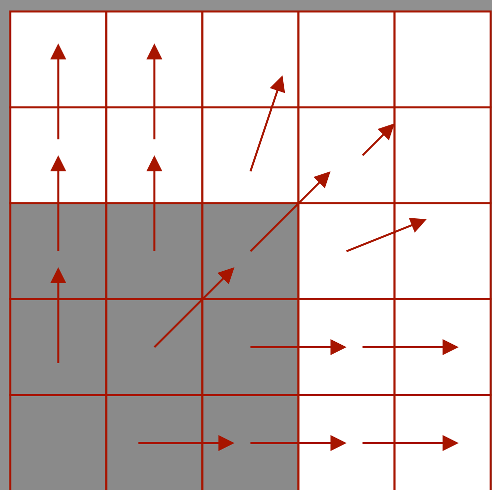


- Use different template orientation at every position
  - At least bilinear interpolation
- Skip positions with no edge
  - i.e. mag ≈ 0

Problem: edge accuracy

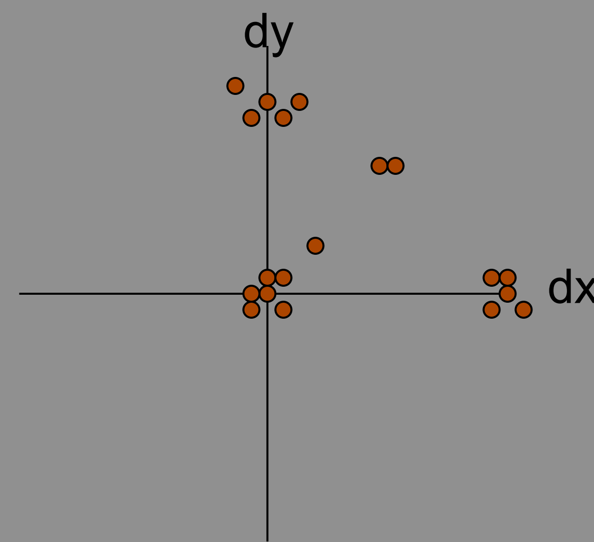
- The orientation of an edge may not be accurate
  - Occlusion
  - Surface marking (smudge)
  - Electronic noise
- Solution: compute dominant edge orientation over a window

Example



5x5 Window

Edges



Dx, Dy Plot

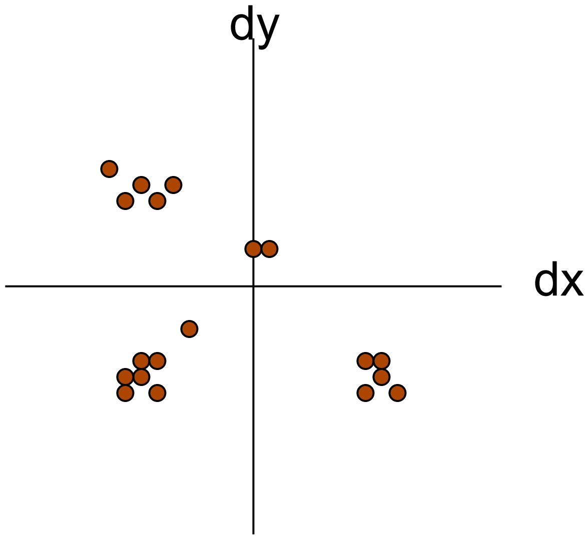
Computing Edge Orientation Dominance

- How do we determine the dominant orientation from a set of [dx, dy] vectors?
- Fit the line that best fits the (dx, dy) points
- Represent edges as a matrix:

$$G = \begin{bmatrix} \partial I / \partial x_1 & \partial I / \partial x_2 & \cdots & \partial I / \partial x_n \\ \partial I / \partial y_1 & \partial I / \partial y_2 & \cdots & \partial I / \partial y_n \end{bmatrix}$$

A general solution…

- Mean center the edge data
- Fit a line the minimizes the squared perpendicular distances



Centered Dx, Dy Plot

Edge Covariance

- Compute the outer product of G with itself:

$$Cov = GG^T = \begin{bmatrix} dx_1 & dx_2 & \cdots & dx_n \\ dy_1 & dy_2 & \cdots & dy_n \end{bmatrix} \begin{bmatrix} dx_1 & dy_1 \\ dx_2 & dy_2 \\ \vdots & \vdots \\ dx_n & dy_n \end{bmatrix} = \begin{bmatrix} \sum_i dx_i^2 & \sum_i dx_i dy_i \\ \sum_i dx_i dy_i & \sum_i dy_i^2 \end{bmatrix}$$

- This matrix is called the *structure tensor*
- What is the semantics of the structure tensor?

Covariance

- Covariance is a measure of whether two signal are linearly related

$$Cov(A,B) = \sum_i (A_i - \overline{A})(B_i - \overline{B})$$

- Note that this is correlation without normalization
- It predicts the linear relationship between the signals
  – i.e. it can be used to fit a line to them

Edge Covariance

- The structure tensor is the covariance matrix of the partial derivatives
  - It tells you the linear relation between the dx and dy values
  - If all the orientations are the same, then dx predicts dy (and vice versa)
  - If the orientations are random, dx has no relation to dy.

Introduction to Principal Components Analysis (PCA)

- We can solve the following:

$$GG^T = R^{-1}\lambda R$$

- Where R is an orthonormal (rotation) matrix and λ is a diagonal matrix with descending values
- What do R and λ tell us?

Eigenvalues and Eigenvectors

- R is a rotation matrix
  - Its rows are axes of a new basis
  - The 1st row (eigenvector) is the best fit direction
    - i.e. the direction of greatest covariance
  - The 2nd eigenvector is orthogonal to the 1st.
- λ contains the eigenvalues
  - The eigenvalues are the <u>covariance</u> in the directions of the new bases
- The closed form equation simply computed the cosine of the first eigenvector

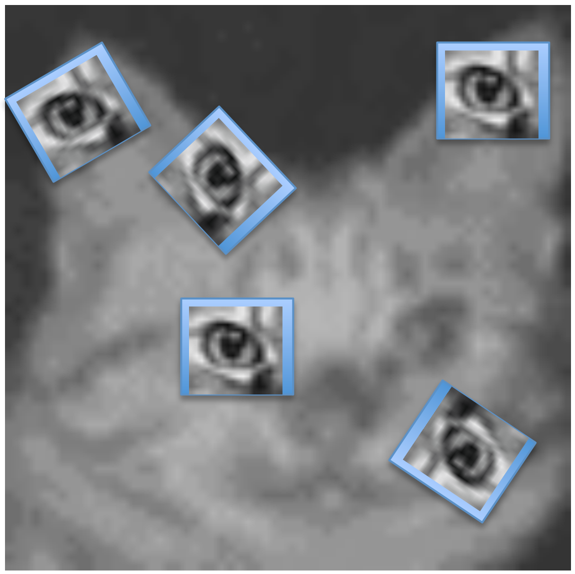
The Structure Tensor

- The structure tensor is the outer product of the partial derivatives with themselves:

$$C = \begin{bmatrix} \sum_i dx_i^2 & \sum_i dx_i dy_i \\ \sum_i dx_i dy_i & \sum_i dy_i^2 \end{bmatrix}$$

- Consider the Eigenvalues
  - Both near zero => no edge (image is locally flat)
  - One large, one near zero => edge
  - Both large => a strong corner

Back to Rotation-Free Correlation



- For every source window:
  - Calculate the edge covariance matrix
  - Find the first eigenvector
    - Skip if 1st eigenvalue is too small
  - Rotate the template to match
  - Correlate

Structure Tensor Eigenvalues

- The structure tensor summarizes the edge information in an image window
- If both eigenvalues are small, the window is a roughly flat surface
  - Not good for matching, good for grouping
- If one eigenvalue is small, the window contains an edge
  - Orientation is reliable
  - Position of match is not (aperture effect)
- If both eigenvalues are larg, the window contains a corner
  - Orientation is one of two
  - Position matches are reliable (good points to match)

Example Points from 2 Images