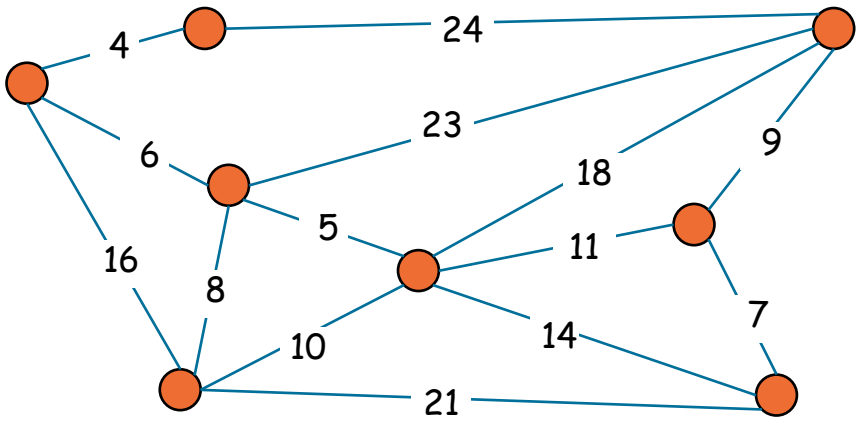
Minimum Spanning Trees
Shortest Paths

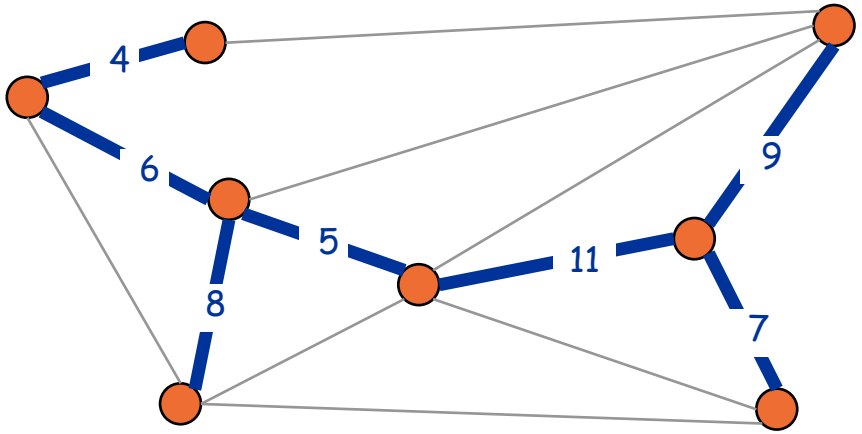
Cormen et. al. VI 23,24

Minimum Spanning Tree

Given a set of locations, with ***positive*** distances to each other, we want to create a sub-graph that connects all nodes to each other with the minimum sum of distances.



$G = (V, E)$



$\Sigma_{e \in T} c_e = 50$

Then that sub-graph is a tree, i.e., has no cycles.

WHY?

If there is a cycle, we can take one edge out of the cycle and still connect all nodes. (Repeat if there are more cycles.)

Applications

MST is fundamental problem with diverse applications.

- Network design.
  - telephone, electrical, hydraulic, TV cable, computer, road
- Approximation algorithms for NP-complete problems.
  - TSP
- ***Cluster analysis.***

Minimal or Minimum Spanning Tree?

1. Minimum is a (singular) noun or adjective, minimal is an adjective
2. Minimum is unique, minimal is when we are not sure
3. Minimum implies that the amount is (relatively) small: Minimum Spanning Tree.

Three Greedy Algorithms for MST

Kruskal's algorithm. Start with $T = \phi$. Consider edges in ascending order of cost. Add edge e to T unless doing so would create a cycle.

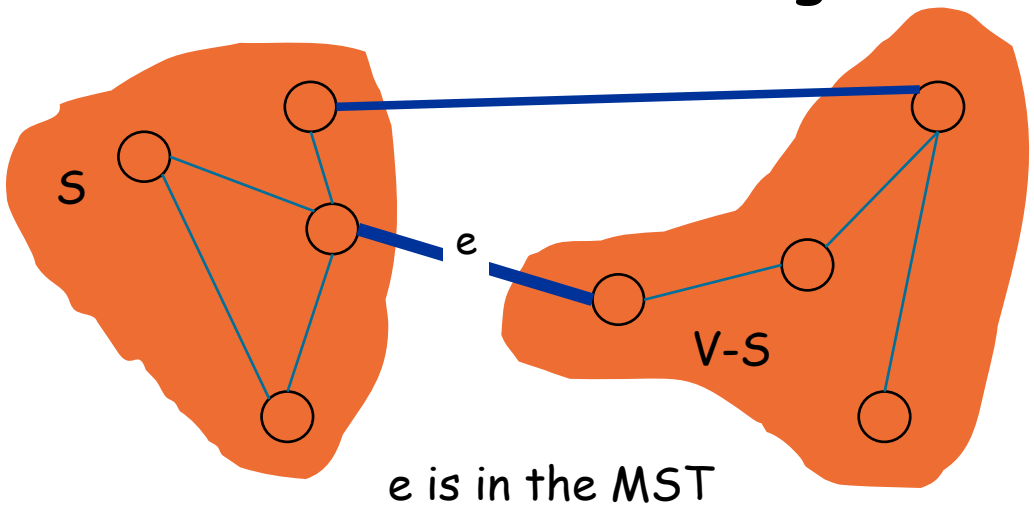
Reverse-Delete algorithm. Start with T = E. Consider edges in descending order of cost. Delete edge e from T unless doing so would disconnect T.

Prim's algorithm. Start with some node s and greedily grow a tree T from s. At each step, add the cheapest edge e to T that has exactly one endpoint in T, i.e., without creating a cycle.

The cut property

Simplifying assumption. All edge costs are distinct.
In this case the MST is unique. In general it is not.

Cut property. Let S be a subset of nodes, S neither empty nor equal V, and let e be the minimum cost edge with exactly one endpoint in S. This is called a *light* edge of the cut.
Then the MST contains e. The cut property establishes the correctness of MST algorithm.



e is in the MST

If multiple equal minimum cost edges, just pick one

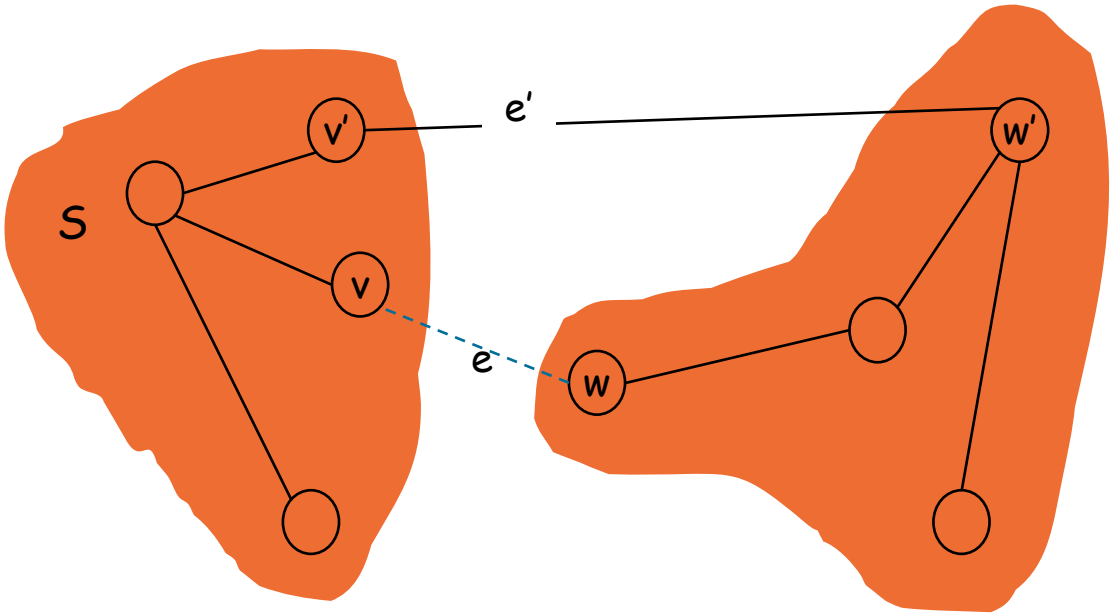
The cut property

Cut property. Let S be a subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then the MST T contains e.

Proof. ***Exchange Argument.***

If e =(v,w) is the only edge connecting S and V-S it must be in T.

Else, there is another edge e'= (v' ,w') with $c_{e'} > c_e$ connecting S and V-S. ***Assume*** e' is in the MST, and not e. Adding e to the spanning tree creates a cycle, then taking out e' out removes the cycle creating a new spanning tree with lower cost. ***Contradiction.***

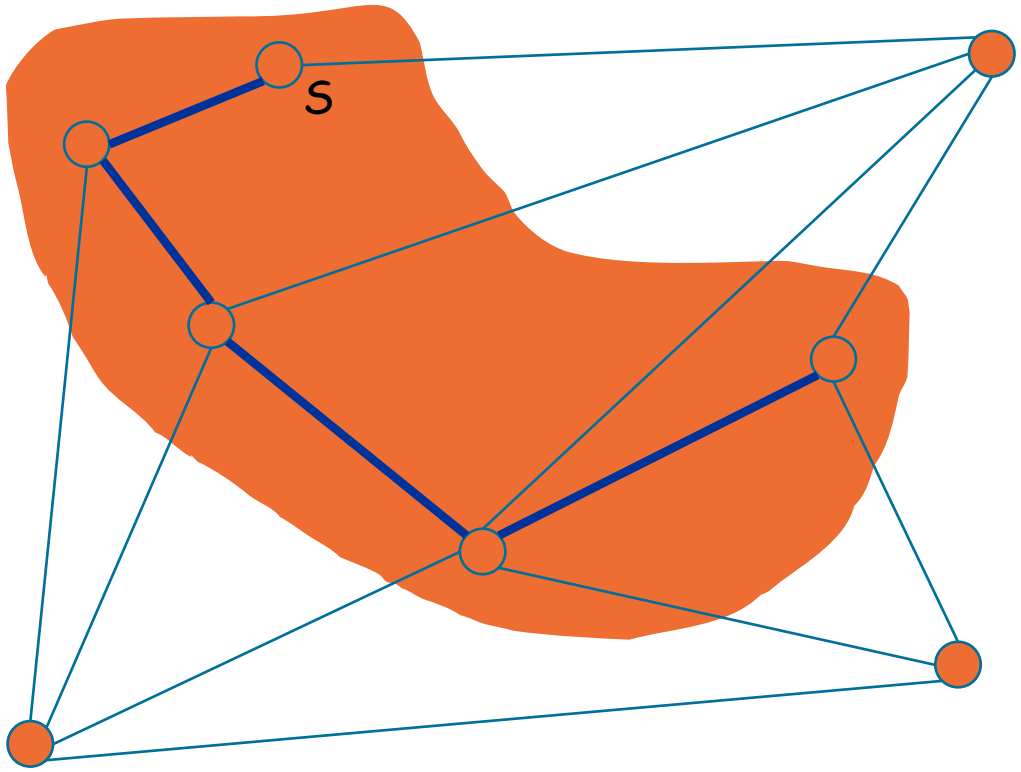


Remember CS220:
if we add an edge to a tree we get a cycle,
if we take any edge out of that cycle we get a tree again.

Prim's Algorithm

Prim's algorithm. [Jarník 1930, Prim 1957, Dijkstra 1959]

- Initialize S = any node.
- Apply cut property to S: add min cost edge (v, w) where v is in S and w is in V-S, and add w to S.
- Repeat until S = V, i.e., greedily growing the MST.

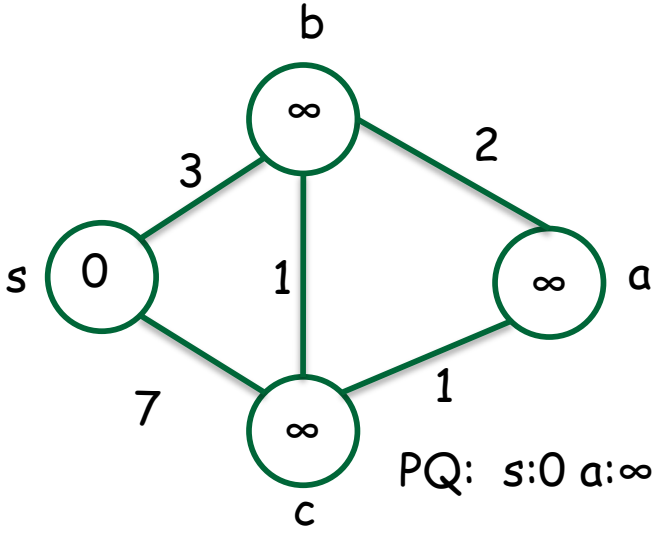


Prim's algorithm: Implementation

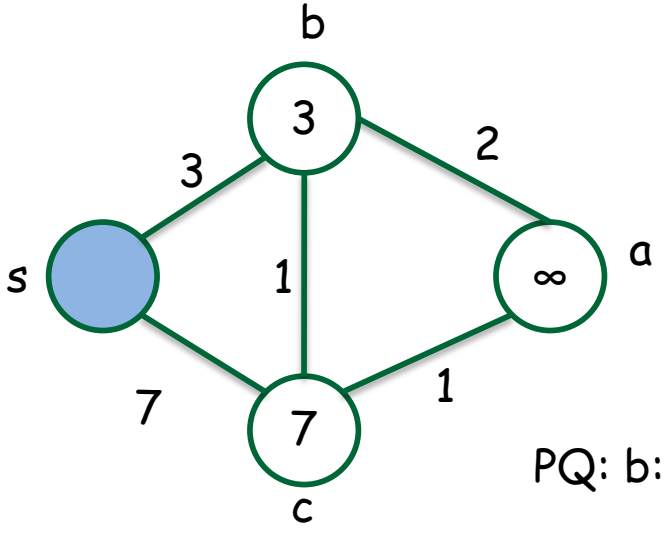
- Maintain set of explored nodes S.
- For each **unexplored** node v, maintain attachment cost a[v] = cost of cheapest edge v to a node in S.

```
Prim(G,s)
    foreach (v ∈ V)
        priority a[v] ← ∞
    a[s]= 0
    priority queue Q = {}
    foreach (v ∈ V) insert v onto Q (key: a[v] )
    set S ← {}
    while (Q is not empty) {
        u ← delete min element from Q
        S ← S ∪ { u }
        foreach (edge e = (u, v) incident to u)
            if ((v ∉ S) and (c_e < a[v]))
                a[v] = c_e // and maintain queue
```

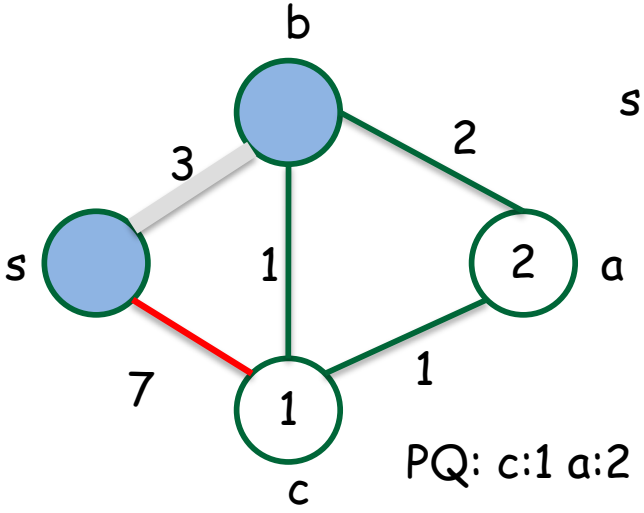
Prim: DO IT, DO IT!



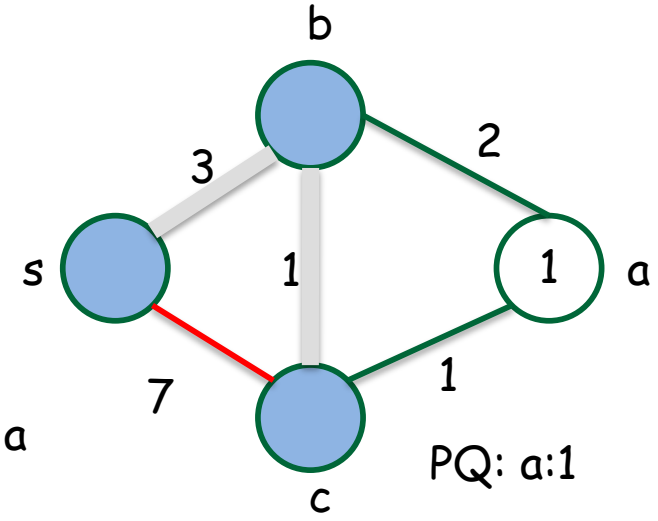
PQ: s:0 a:∞ b:∞ c:∞



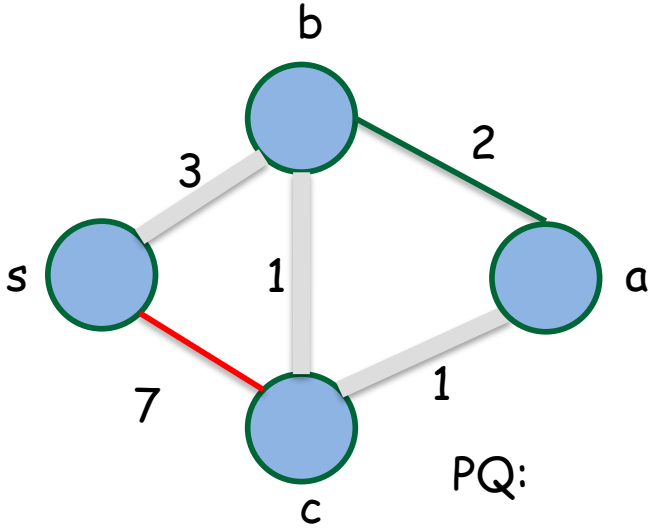
PQ: b:3 c:7 a:∞



PQ: c:1 a:2

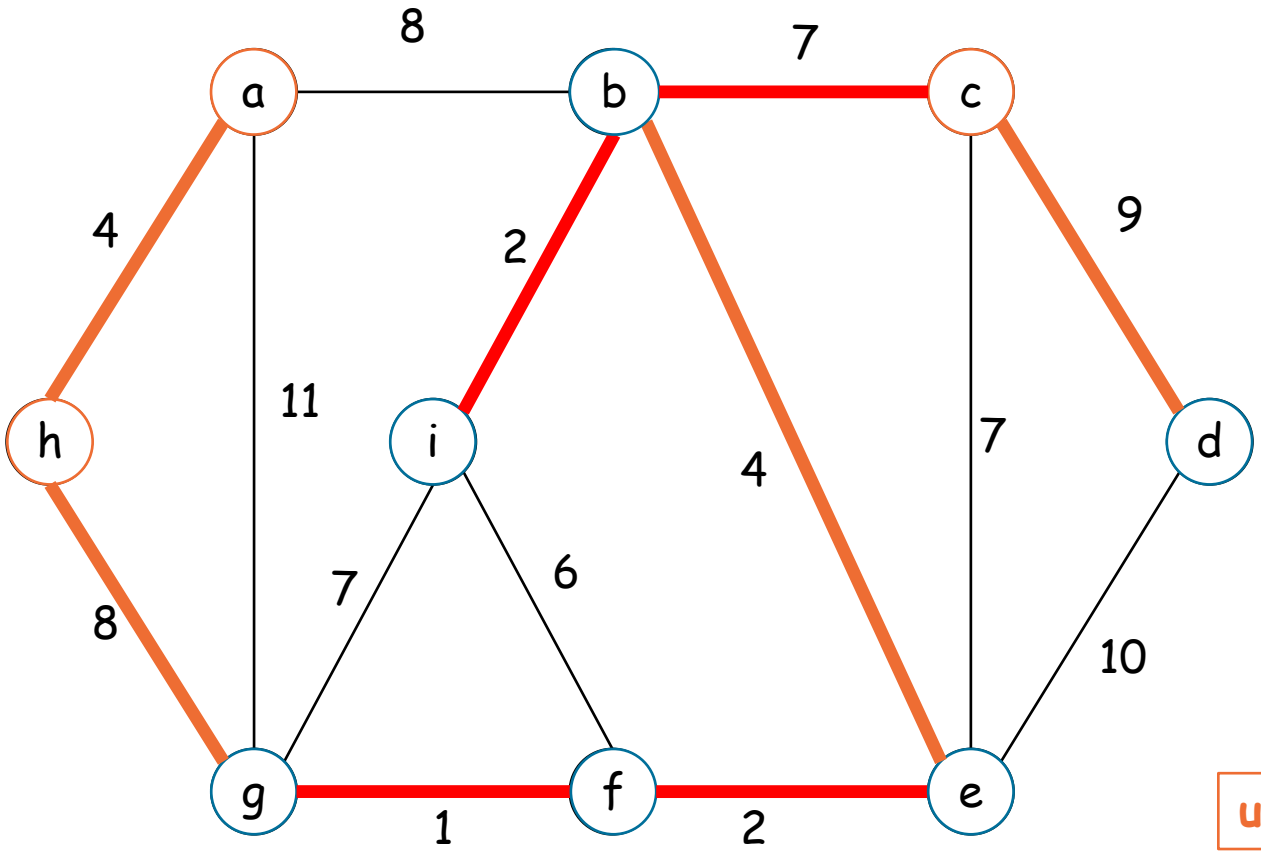


PQ: a:1



PQ:

Let's do the Prim again, starting at d



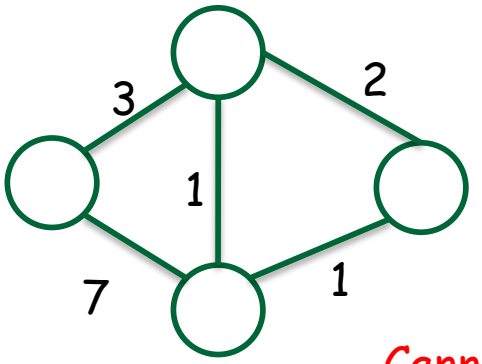
unique?

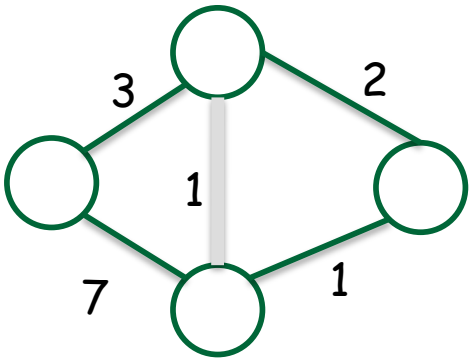
{(d,c),(c,b), (b,i), (b,e), (e,f), (f,g), (g,h), (h,a) }

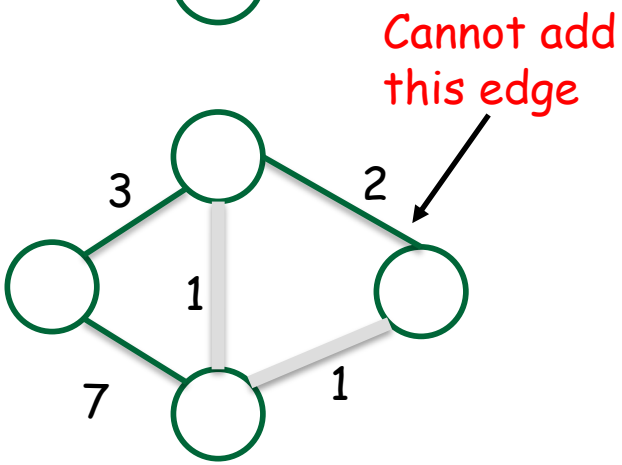
Kruskal's algorithm [Kruskal, 1956]

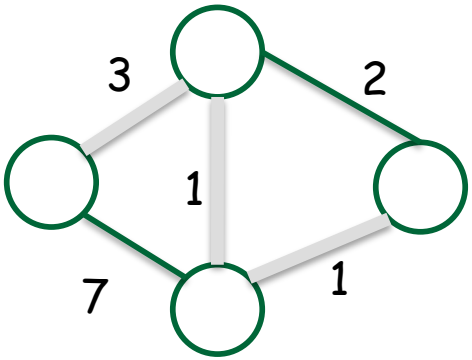
Kruskal:

Consider edges in ascending order of weight. Add edge unless doing so would create a cycle.

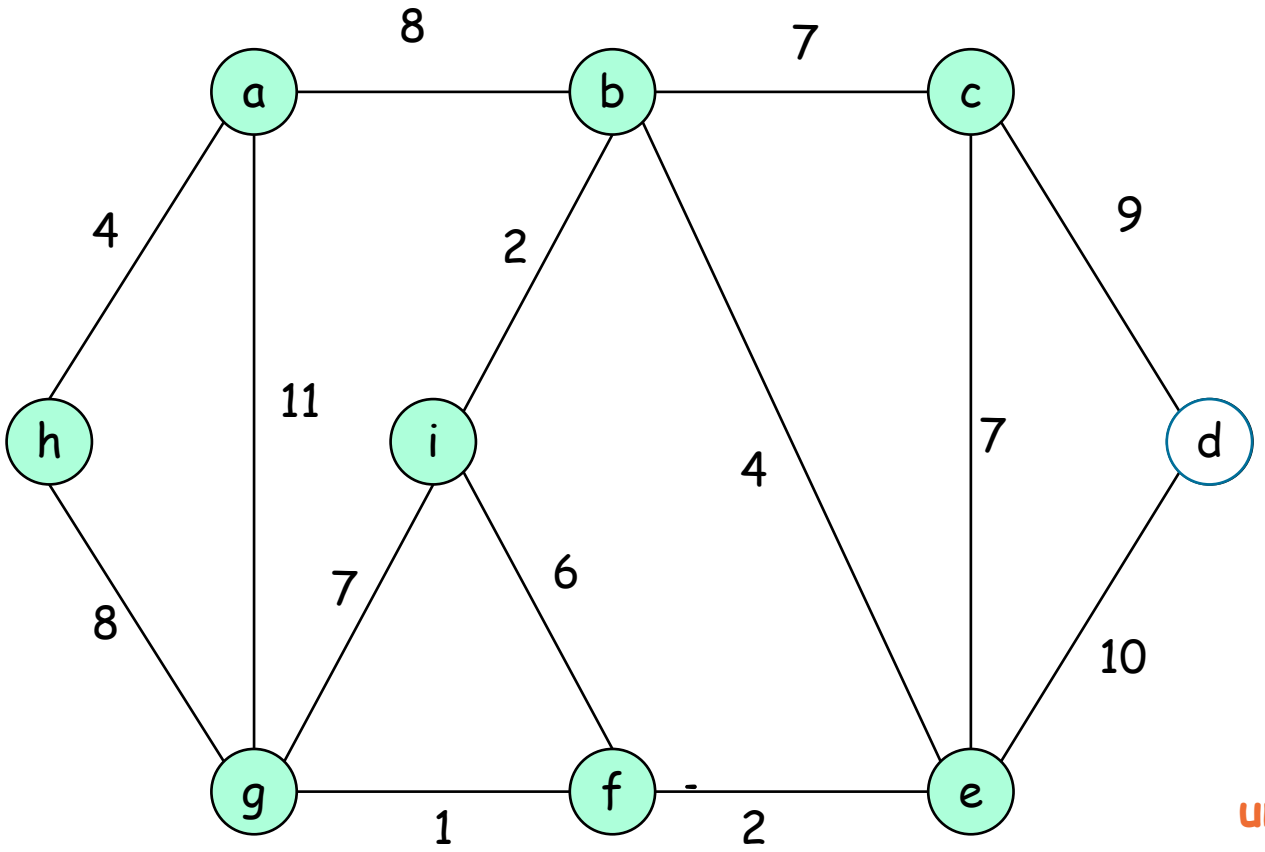








Let's do Kruskal's algorithm

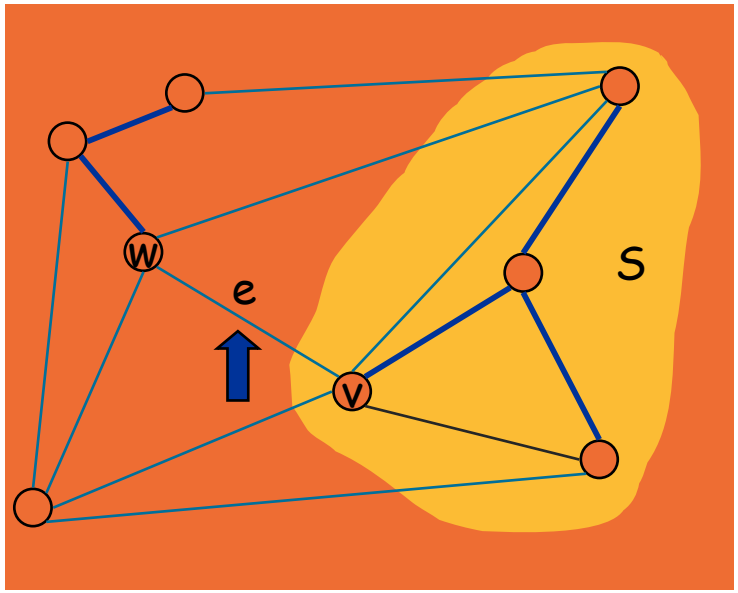


unique?

Kruskal works

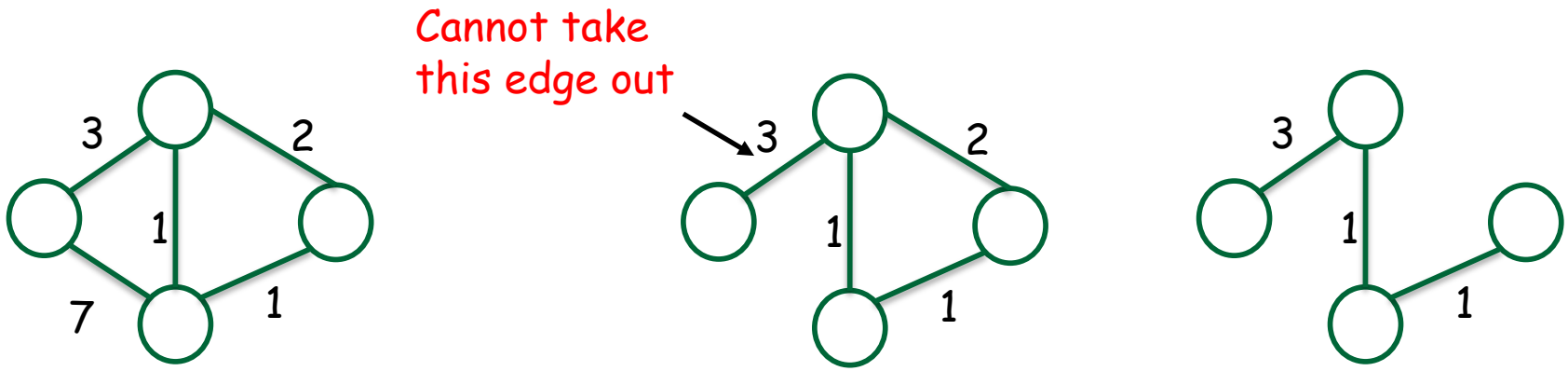
1 **Spanning Tree:** Kruskal keeps adding edges until all nodes are connected, and does not create cycles, so produces a spanning tree.

2. **Minimum Spanning Tree:** Consider e=(v, w) added by Kruskal. S is the set of nodes connected to v just before e is added; v is in S and w is not (otherwise we created a cycle). Therefore e is the cheapest edge connecting S to a node in V-S, and hence, e is in any MST (cut property).



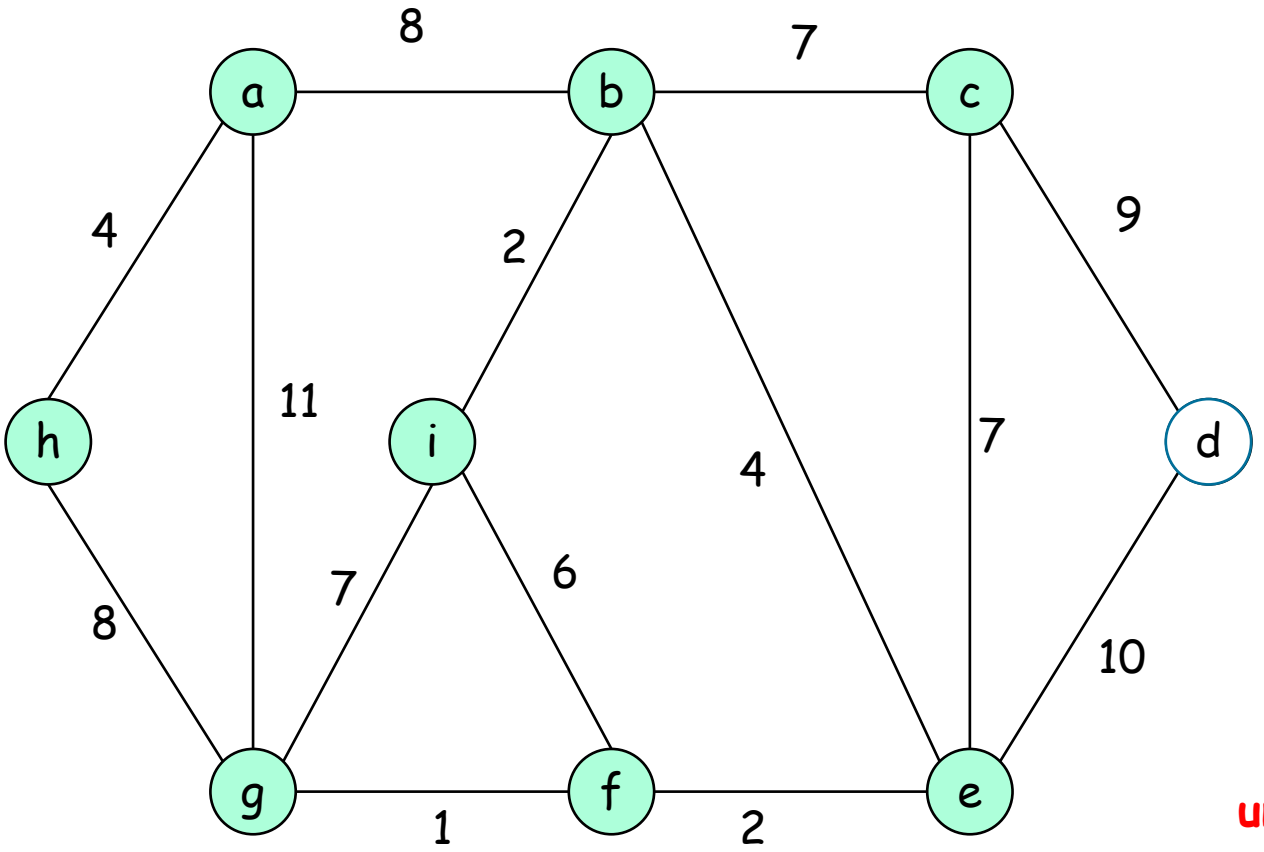
Reverse-Delete algorithm

Start with T = E. Consider edges in descending order of cost. Delete edge e from T unless doing so would disconnect T.



Is it always safe to remove e, i.e. could e be in an MST?

Let's do the Reverse Delete algorithm



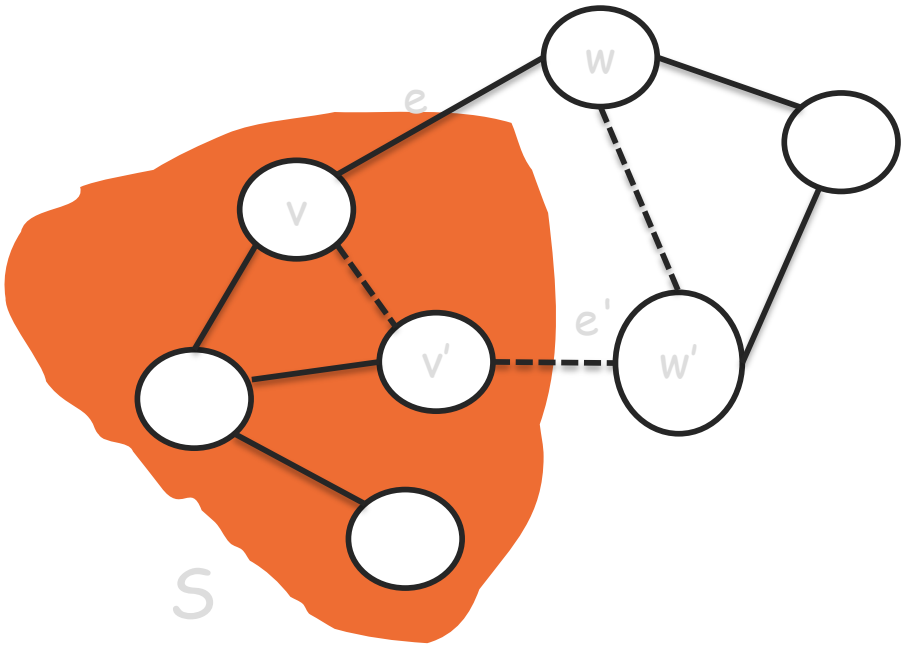
unique?

Safely removing edges

Cycle property. Let C be any cycle in G, and let e be the max cost edge belonging to C. Then e doesn't belong to any MST of G.

Let T be a spanning tree that contains max edge e=(v,w). Remove e; this will disconnect T, creating S containing v, and V-S containing w.

C–{e} is a path P. Following P from v will at some stage cross S into V-S by edge e' with lower cost than e, so T - {e} + {e'} is again a spanning tree and its cost is lower than T, so T is not an MST.



Shortest Paths Problems

Given a **weighted** graph G=(V,E) find the shortest path

- path length is the sum of its edge weights.

The shortest path from u to v is ∞ if there is no path from u to v.

Variations of the shortest path problem:

1) **SSSP** (Single source SP): find the SP from some node s to all nodes in the graph.

2) **SPSP** (single pair SP): find the SP from some u to some v.
We can use 1) to solve 2), also there is no more efficient algorithm for 2) than that for 1).

3) **SDSP** (single destination SP) can use 1) by reversing its edges.

4) **APSP** (all pair SPs) could be solved by |V| applications of 1), but there are other approaches (cs420).

Dijkstra SSSP

Dijkstra's (Greedy) SSSP algorithm only works for graphs with only positive edge weights.

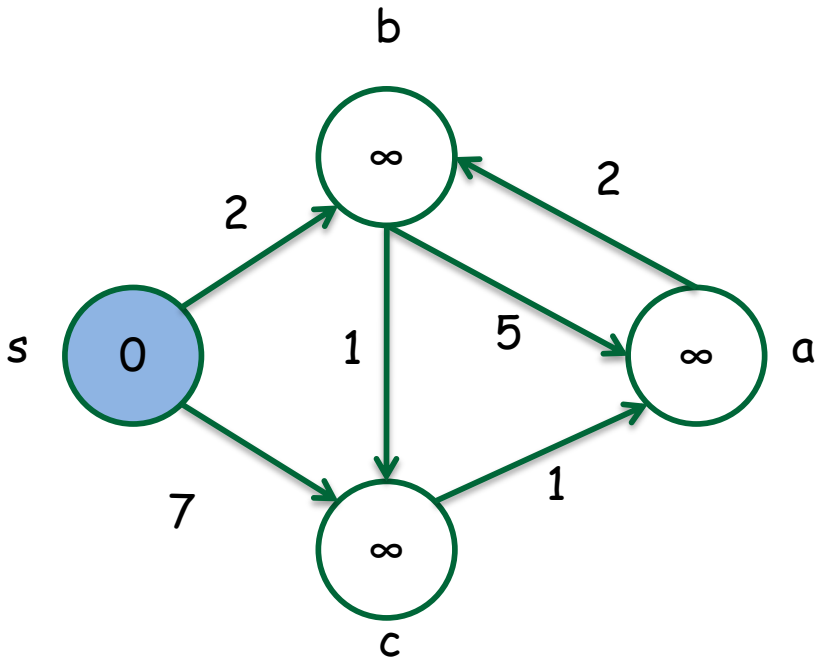
S is the set of explored nodes. For each u in S, d[u] is a distance.
Initialize: S = {s} the source, and d[s]=0, for all
other nodes v in V-S, d[v]= ∞
while S≠V:
  select a node v in V-S with at least one edge
  from S, for which **d'[v]=min$_{e=(u,v),u \text{ in } S}$d[u]+w$_e$**
  add v to S (S=S+v)
  d[v]=d'[v]

What does d' represent?

the minimum **path** length extending with one edge out of S

To compute the actual minimum paths, maintain an array p[v] of predecessors. When d[v] is set to d'[v], set p[v] to u.

Notice: Dijkstra is very similar to Primm, but where Dijkstra minimizes path lengths, Prim minimizes edge lengths.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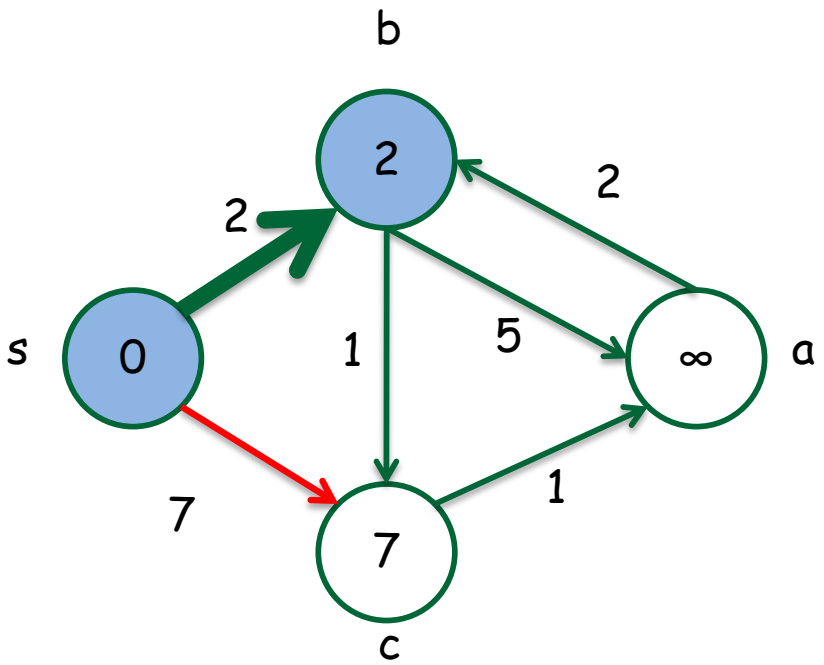
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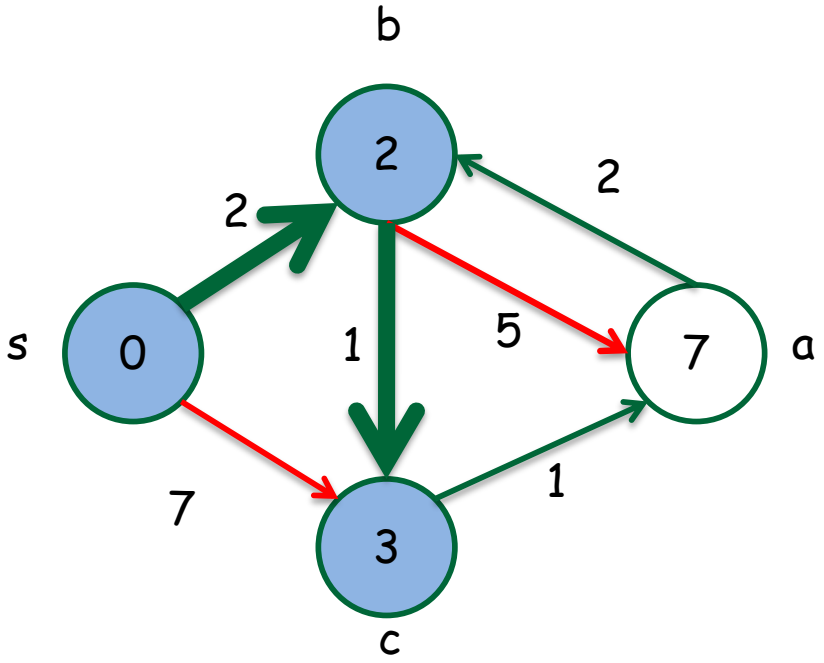
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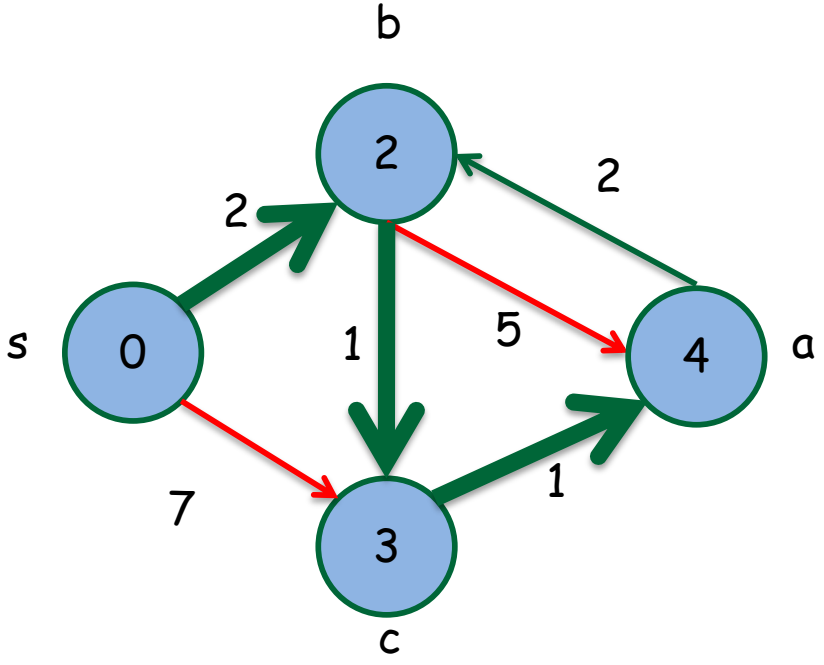
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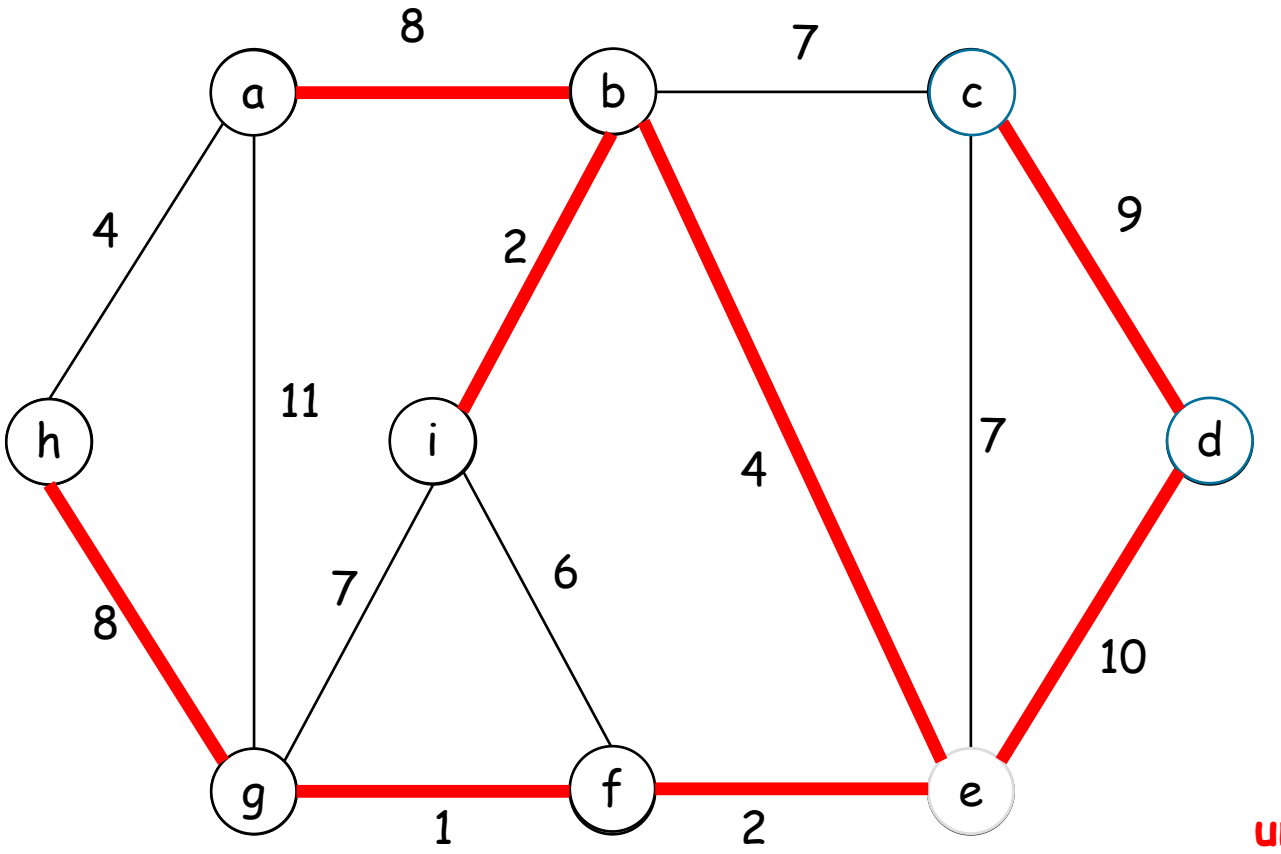
Let's do Dijkstra, starting at d

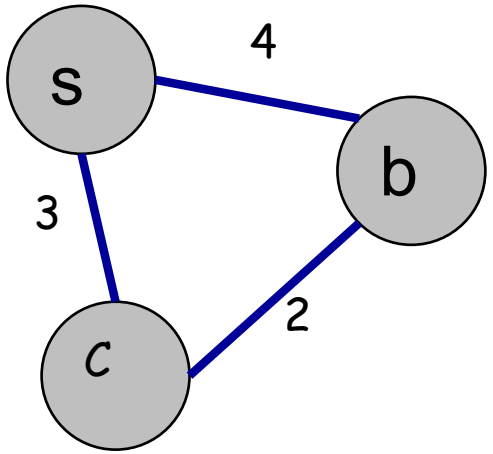


unique?

In this case yes, because all path lengths are different
But in general, multiple path lengths can be equal, and thus can lead to different choices.

Does Dijkstra's algorithm lead to a Minimum Spanning Tree?

Can you create a counter example?



Shortest paths from S?

Minimum Spanning Tree?

Formulate the difference between Prim and Dijkstra

Dijkstra works

For each u in S, the path $P_{s,u}$ is the shortest (s,u) path

Proof: by induction on the size of S

Base: |S| = 1 d[s]=0 OK

Step: Suppose it holds for |S|=k>=1, then grow S by 1 adding node v using edge (u,v) (u already in S) to create the next S.
Then path $P_{s,u,v}$ is path $P_{s,u}$+(u,v), and is the shortest path to v

WHY? What are the "ingredients" of an exchange argument?
What are the inequalities?

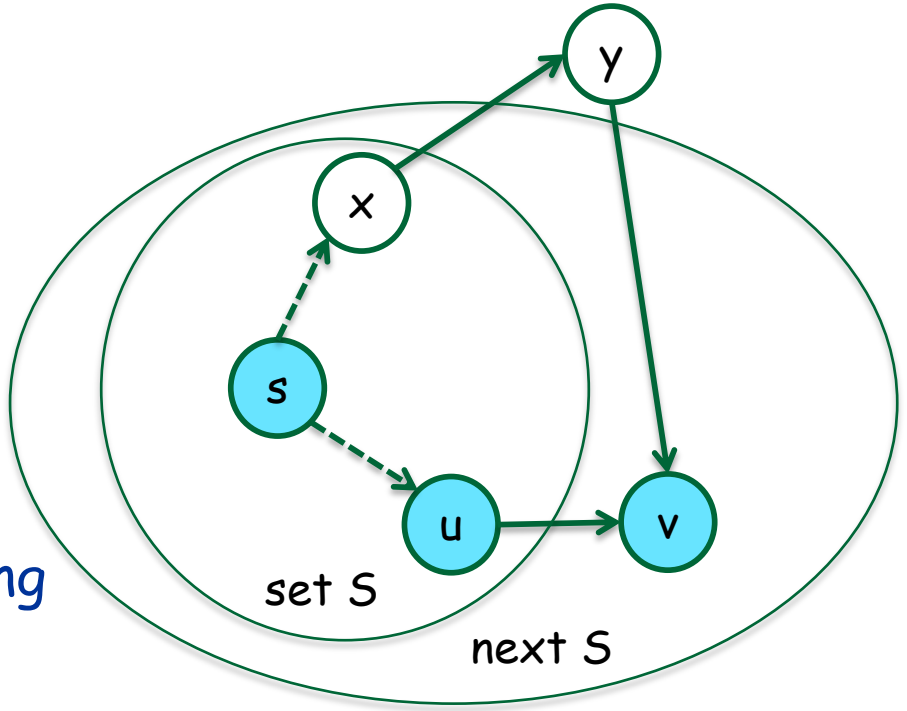
Greedy exchange argument

Assume there is **another path P from s to v**. P leaves s with edge (x,y). Then the path P goes from s to x to y to v.
What can you say about P: s →* x → y compared to $P_{s,u,v}$? How does the algorithm pick $P_{s,u,v}$? Why does it not work for negative edges?

P from s to y is at least as long as $P_{s,u,v}$ because the algorithm picks the shortest extension out of S.

Hence the path
P: s →* x → y →* v is at least as long as
$P_{s,u,v}$: s →* u → v

This would not work if w(y,v) <0



Corner case?

s→x→y = s→u→v AND y = v