CS 320
Algorithms: Theory and Practice
Runtime Analysis: Big-Oh

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Week 2
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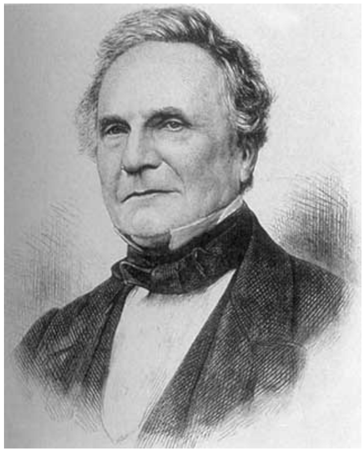
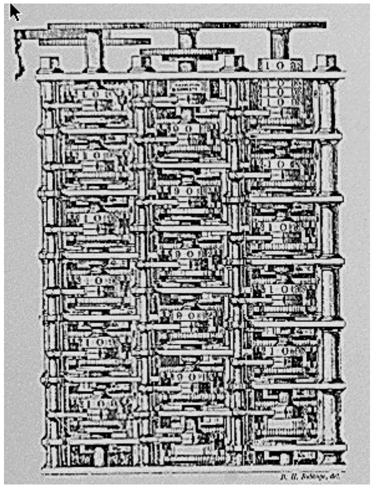
Runtime Analysis

As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise - By what course of calculation can these results be arrived at by the machine in the shortest time? - *Charles Babbage*



Charles Babbage (1864)



Analytic Engine (schematic)

Outline: four topics

- Algorithm time complexity
- Plotting data and the *function clubs*
- Digression: line of sight algorithm
- A survey of common running times

Algorithm Time Complexity

Algorithm Time Complexity

- How do we measure the *complexity* (time, space requirements) of an algorithm?
  - As a function of its input *size* (an integer, n) denoting:
    - Number of inputs (e.g., sorting)
    - Number of bits to represent the input (e.g., primality)
    - Sometimes multiple parameters, e.g., knapsack
      - Number of objects, n
      - Knapsack capacity, C

We want to determine the running time as a function of problem sizes, and analyze them asymptotically

How to measure time?

- Seconds/nano-seconds?
  - *No, too specific & machine dependent*
- Number of instructions executed?
  - *No, still too specific & machine dependent*
- # of code fragments that take constant time?
  - *Yes*
- What kind of fragments/instructions?
  - *Arithmetic operations,*
    memory accesses,
    finite combinations of these

How to measure space?

- Bits?
  - *Too detailed, but sometimes necessary (e.g., knapsack capacity)*
- Integers?
  - *Nicer, but dangerous –* we can code a whole program in a single *arbitrary sized* integer, so we have to be careful about the size. Better to use machine words i.e, fixed size (e.g., 64, collections of bits)

Worst/average case time

- A bound on the *maximum possible* running time of the algorithm of inputs of size n
  - Usually captures the notion, but may be an overestimate
- Average case
  - More accurate but difficult – need to describe what is the range of inputs, and what is the distribution, statistical analysis. Let I be the set of inputs, and P_i and C_i be the probability and computation time of input i

$$\sum_{i \in I} P_i C_i$$

  - Often a constant factor of worst case time

Same considerations for space and other measures.

Why it matters

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	4 sec
$n = 30$	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	18 min	10^{25} years
$n = 50$	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	11 min	36 years	very long
$n = 100$	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	$<$ 1 sec	$<$ 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	$<$ 1 sec	$<$ 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	$<$ 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Tractable = polynomial time

- For many problems, there is a natural, but likely naïve, brute force search algorithm that checks every possible solution
  - Enumerating such solutions is usually an exponential function of n (recall counting from CS220).
  - Hence naïve
- *Definition*: an algorithm is said to be polynomial time if there exist positive constants c, and d, such that on any input of size n, the running time is bounded by $c\, n^d$

What about an algorithm whose running time is $c\, n \lg n$?

Justification/issues

(Why) is the distinction important?

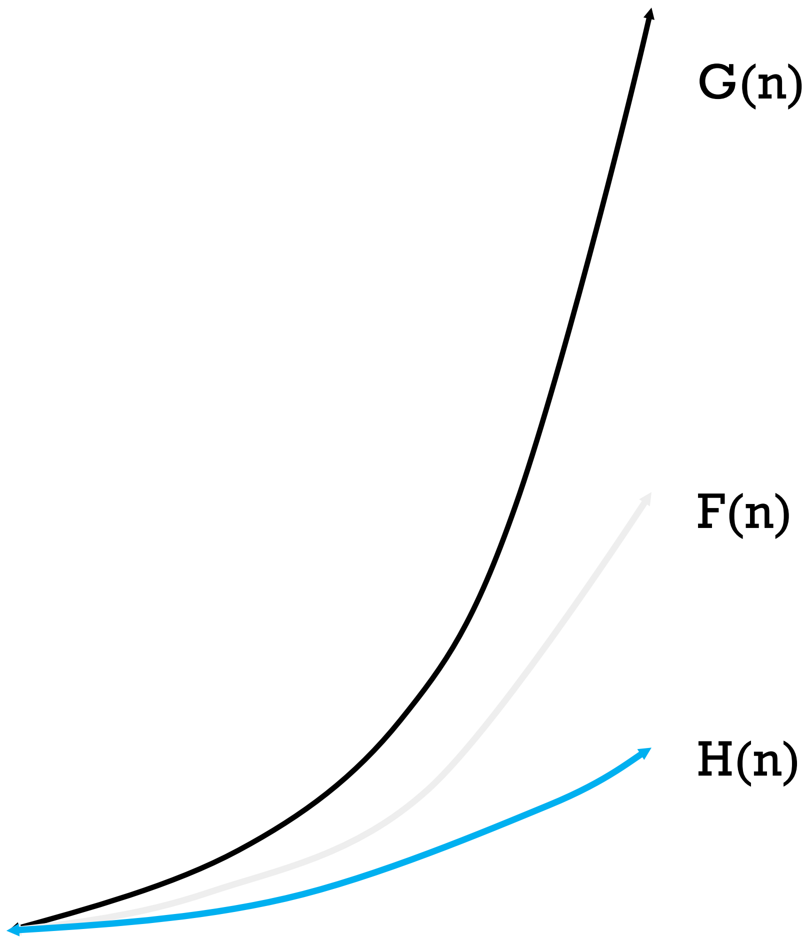
- One the one hand, a polynomial function like $6.03 \times 10^{23}\, n^{20}$ is polynomial, it is too large in practice (e.g., for $n = 10$)
- On the other hand, some algorithm whose worst-case execution time is exponential *behave much better* in practice because the worst-case instances are (seem to be) rare
  - Simplex method for solving *linear programming*

So why?

- In practice, the polynomials have a low degree and coefficients
- The difference between the polynomial-exponential barrier reveals interesting and crucial structure of the problem

Asymptotic growth rates

- We are building mathematical functions that model the execution time (or other properties) of programs and algorithms.
- Need a mechanism to *compare* them.
  - How do we compare numbers? Using the relations: $<, >, \leq$, and $\geq$
  - Partial/total orders
- The Big-Oh, Big-Omega and Big-Theta notation (introduced in CS 220) is such an order relation. Here, $f \lessdot g$ means that f grows *slower* than g (and also that g grows *faster* than f). We may also use $g \gtrdot f$. So the following claims mean the same thing
  - $f(n) \lessdot g(n)$ You may see the symbol $\succcurlyeq$ if the tool doesn't have the right font.
  - $g \gtrdot f$ or $g \succcurlyeq f$
  - $f = O(g)$
  - $f(n) = O(g(n))$
  - $g(n) = \Omega(f(n))$
- Often, one of the functions is our (complicated) model $T(n)$ and the other is a simpler function (e.g., a monomial)



$F(n)$ is $O(G(n))$

$F(n)$ is $\Omega(H(n))$

if $G(n) = c\ H(n)$
then $F(n)$ is $\Theta(G(n))$

These measures were introduced in CS220

Basic definitions

A function $T(n)$ is $O(f(n))$ if there exist constants

- $c > 0$, and $n_0 > 0$ such that for all $n \geq n_0$,
  $T(n) \leq c\,f(n)$

- Example: $T(n) = 32n^2 + 16n + 32$.
  - $T(n)$ is $O(n^2)$
  - *ALSO TRUE*:
    - $T(n)$ is $O(n^3)$
    - $T(n)$ is $O(2^n)$
  - Many possible upper bounds for one function! We always look for the best (lowest) upper bound, but it not always easy to establish
  - Which function grows faster?
    - $f(x) = \sqrt[4]{x}$ or
    - $g(x) = \log^2 x$
  - Where is the crossover?

Properties of $\lessdot$, $\gtrdot$ and 0

- Transitivity
  - $f \lessdot g$ and $g \lessdot h$ implies $f \lessdot h$
- Additivity (Additive slowdown)
  - $f \lessdot h$ and $g \lessdot h$ implies $f + g \lessdot h$
- Multiplication by a constant
  - $f \lessdot g$ implies $c \times f \lessdot g$ (and of course $f \lessdot c \times g$ holds by definition)

Lower bounds (convention)

Although Big-Oh and Big-Omega are equivalent, a special need arises when our model $T(n)$ is quantified over *all algorithms to solve the given problem*

- Example: consider the claim that any comparison based algorithm must make at least $c \times n \ \log n$ comparisons, for some constant, c. We say that comparison based sorting is *lower bounded* by $n \lg n$, i.e., that $T(n)$ is $\Omega(n \lg n)$ and we often reserve the Ω notation for this.
- *Problems have lower bounds*
  - A common lower bound is the size of the input itself (any algorithm to solve the problem must read all the inputs)
  - Sometimes we can prove better/tighter lower bounds (e.g., sorting above and searching in structured data (CS 420)

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Tight Bounds

- If $T(n)$ is $\Omega(f(n))$ and $T(n)$ is also $O(f(n))$ we have a tight bound, and we write that $T(n)$ is $\Theta(f(n))$.
- It means that we have *closed the problem*, since the *algorithm* that we have attains the lower bound on the *problem*

Closed and Open Problems

Sorting is a *closed problem*

- It has a lower bound of $n \log n$. We say that *sorting is* $\Omega(n \log n)$
- There are many sorting algorithms whose execution time is $O(n \log n)$ (*see how we use big-Oh to talk about an algorithm*)

Matrix multiplication is an *open problem*

- It is $\Omega(n^2)$. *Why?*
  - The standard algorithm is $O(n^3)$
  - Another well known algorithm is $O(n^{2.376})$ and further improvements reduce the polynomial degree even further

Note: polynomial degree does not have to be integer

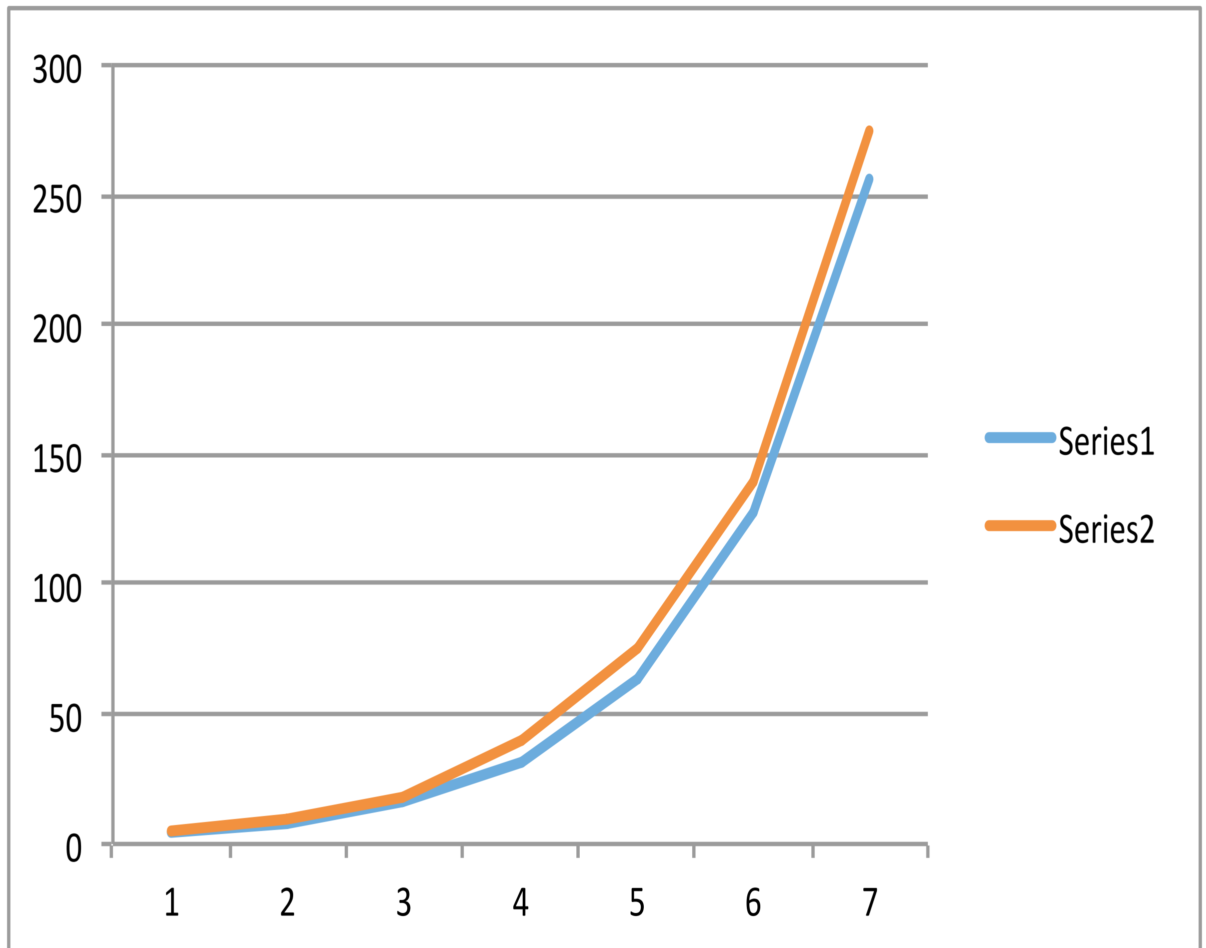
Plotting functions cleanly

Plotting functions cleanly

- In empirical CS (HPC, performance optimization, parallel programming) we plot functions describing the run time (or the memory use) of a program:
  - This can be as a function of the input size (or other parameters like # of processors)
- The functions are usually positive and monotonically increasing
- We are interested in the asymptotic behavior, i.e., $\lim_{n \to \infty} f(n)$
- How should we graph/plot them (e.g., lab report)?

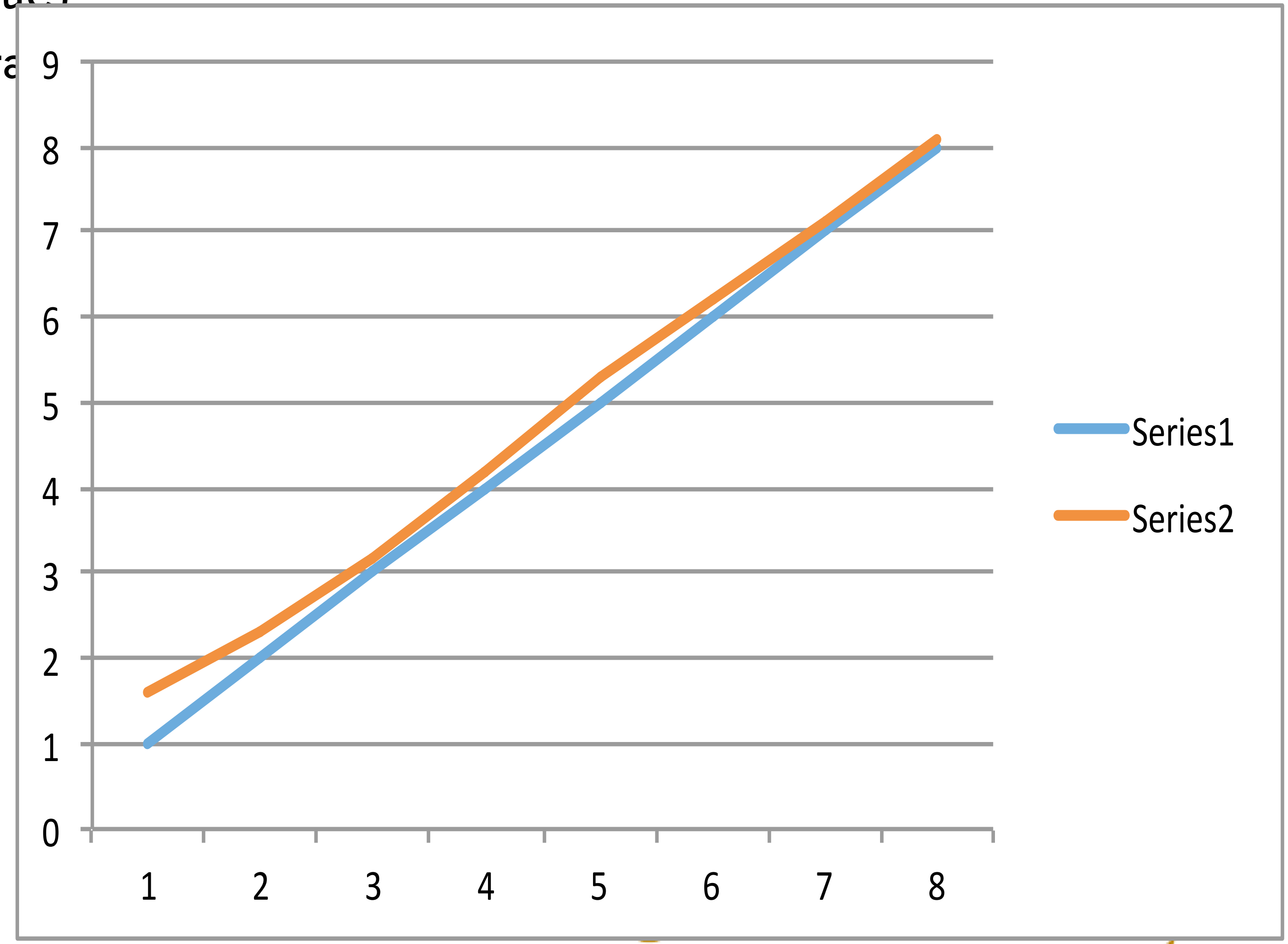
Ungraded Quiz (survey)

- The plot shows the ideal (expected) vs empirical (observed) values. Which one is ideal, and which is "just a bit off?"
  - Series 1 (blue)
  - Series 2 (orange)



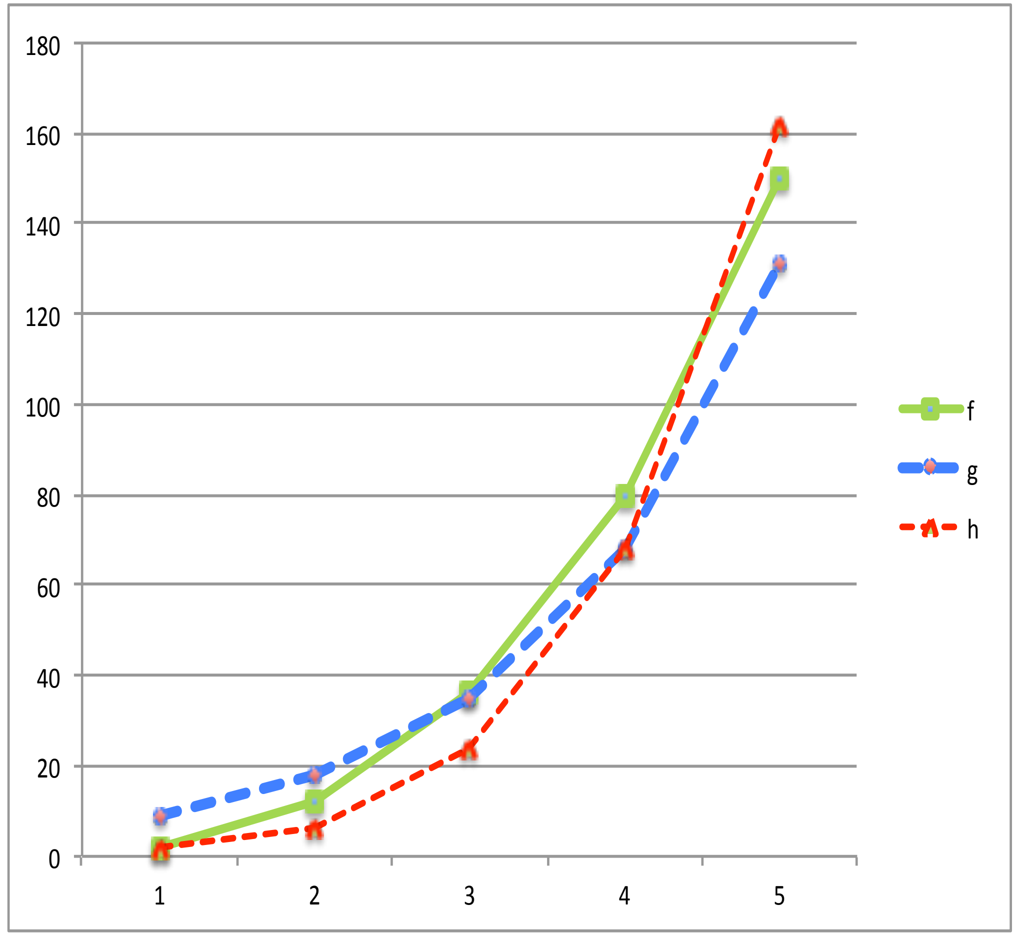
Ungraded Quiz (survey)

- Same question, data is plotted differently.
  - Series 1 (blue)
  - Series 2 (ora



Three functions: f, g and h

n	f(n)	g(n)	h(n)
1	2	9	2
2	12	18	6
3	36	35	24
4	80	68	68
5	150	131	162



- What class of functions are f, g, and h?
  - Polynomial? What degree?
  - Exponential? What base?
- Impossible/hard to tell

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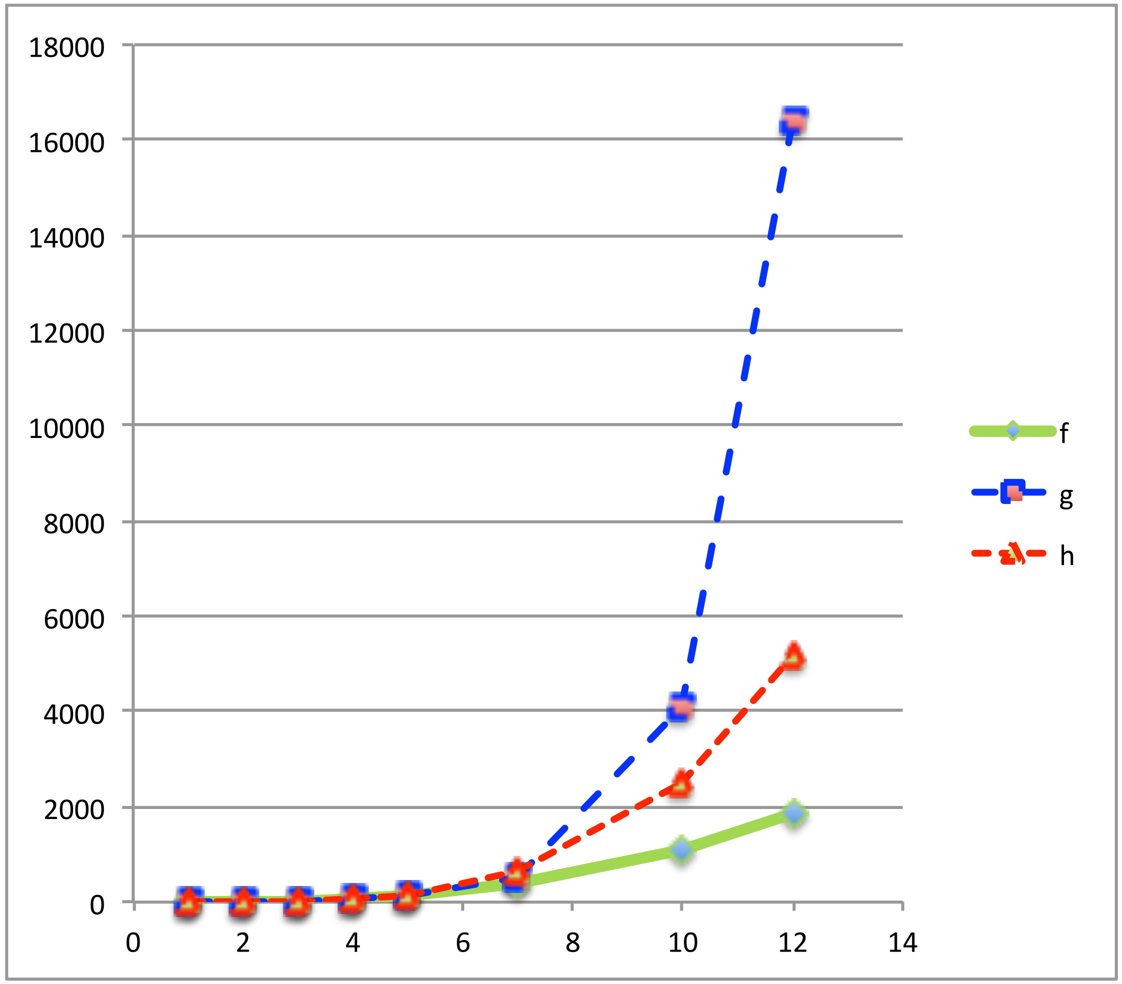
Why is it hard?

- The human visual system is very good at identifying linear (straight line) plots.
- Everything else is approximate.
- Asymptotically increasing functions just "swoosh up," i.e., $\lim_{n \to \infty} f(n) = \infty$
- Not enough range of data in second set of examples here just 1 … 5)

Larger domains for f, g and h

n	f(n)	g(n)	h(n)
1	2	9	2
2	12	18	6
3	36	35	24
4	80	68	68
5	150	131	162
7	400	520	624
10	1100	4106	2510
12	1872	16396	5196



Much better idea now about which function may be polynomial vs exponential? But still

- all is not clear (order, base …)
- h(n) may spike up later…

Straight lines

Visually, we get the most information from *straight lines*!

- We can easily recognize a straight line

  $$y = ax + b$$

  - The *slope* (a) and *y intercept* (b) tells us all.
  - *How to "massage the data" into straight lines.*
  - Change the scale to logarithmic: it turns a *multiplicative* factor into a *shift* (y axis crossing b) , and an *exponential* into a *multiplicative factor* (slope a)

Four exponential functions

$y = 2^n$ $\boxed{\log_{10}(y)} = \boxed{n} \log_{10} 2$ **linear in *n***

$y = 3^n$ $\boxed{\log_{10}(y)} = \boxed{n} \log_{10} 3$

the slope is the (log of the) base of the exponent

$y = 6 \times 3^n$ $\boxed{\log_{10}(y)} = \boxed{n} \log_{10} 3 + \log_{10} 6$

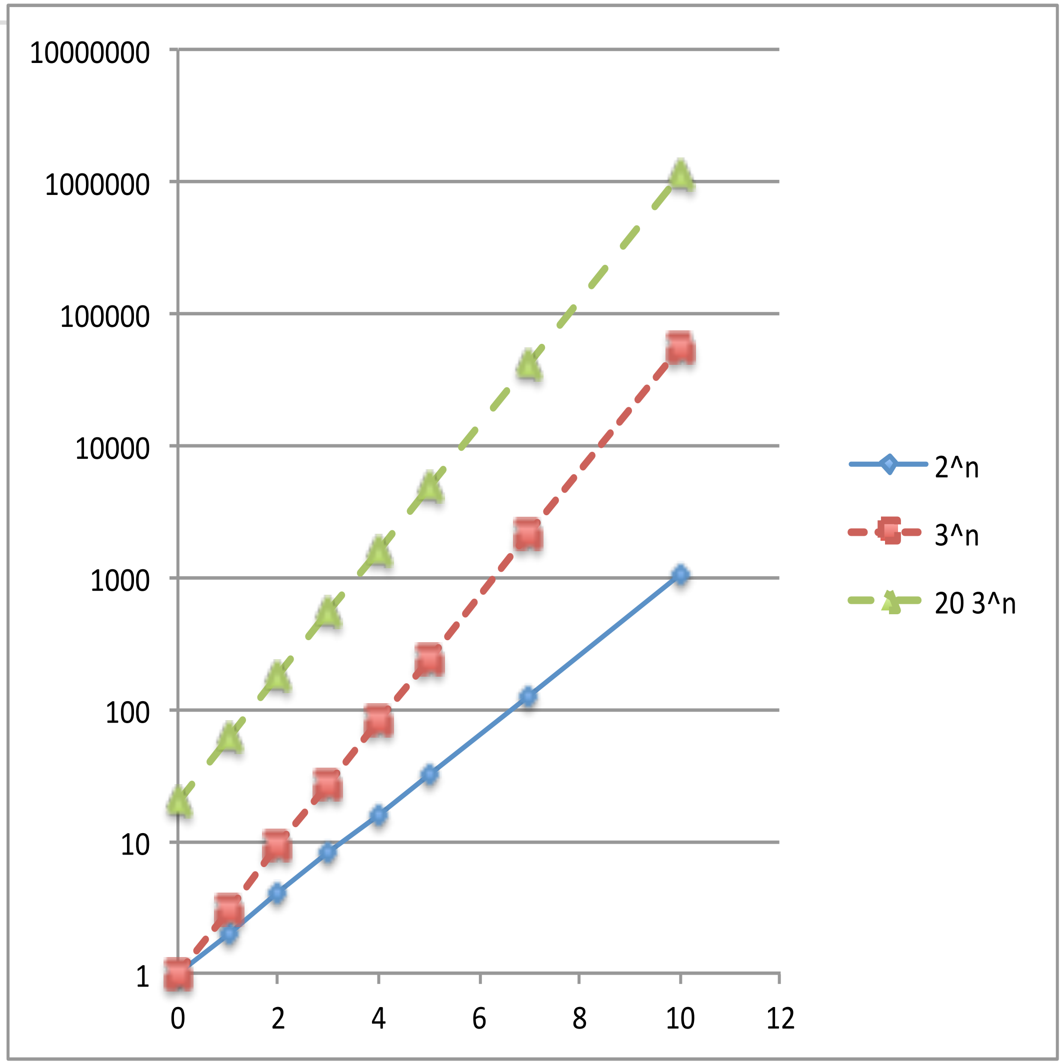
6 shifts up (in log scale)

$y = 3^n/5$ $\boxed{\log_{10}(y)} = \boxed{n} \log_{10} 3 - \log_{10} 5$

5 shifts down

Use a semi-log plot

n	2^n	3^n	$20*3^n$
0	1	1	20
1	2	3	60
2	4	9	180
3	8	27	540
4	16	81	1620
5	32	243	4860
7	128	2087	41740
10	1024	56349	1126980



semi-log plot:
    y–axis on log scale
    x-axis linear
angle: base
shift: multiplicative factor

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What about polynomials?

What is the logarithm of a polynomial (actually a monomial)?

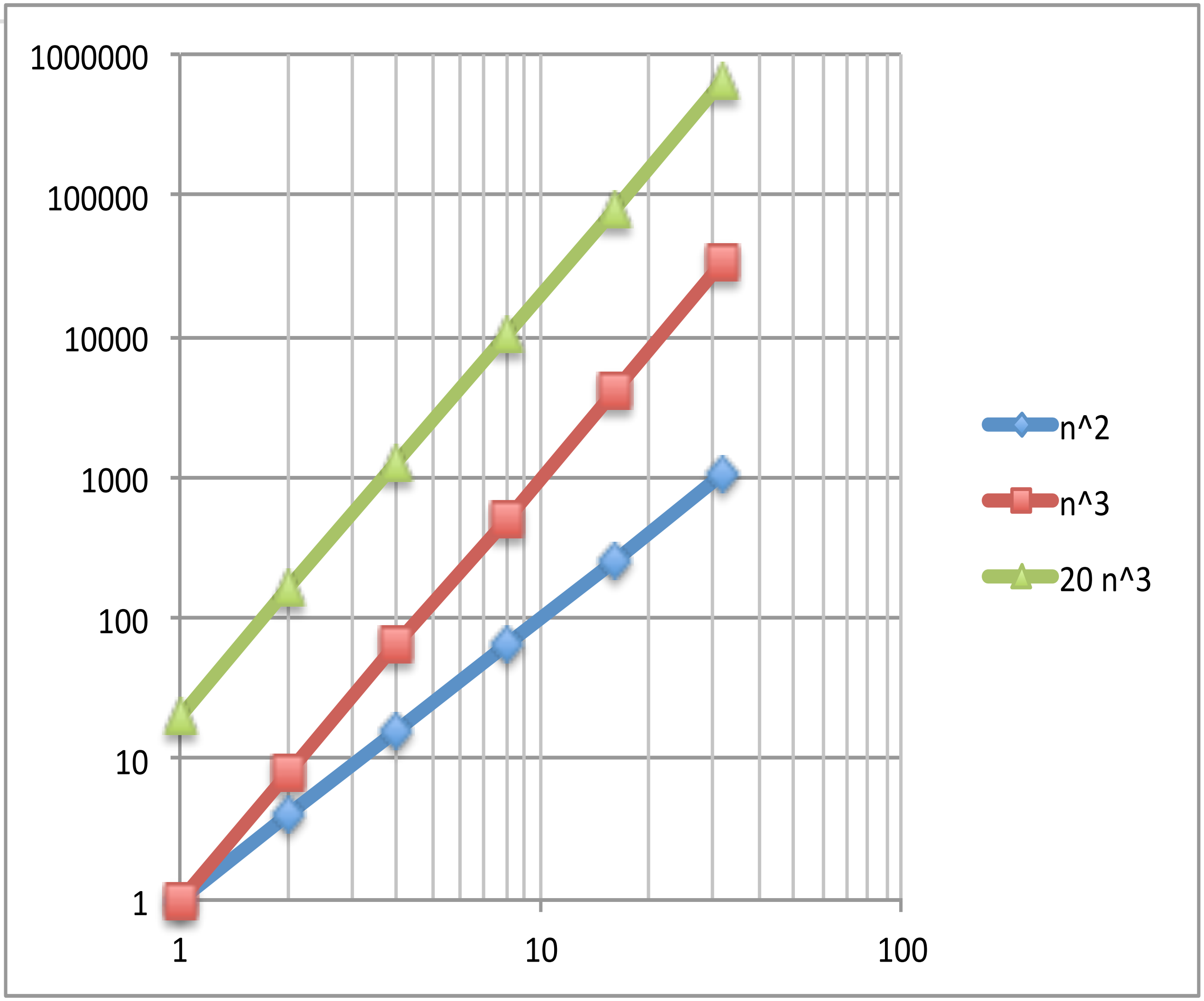
- $y = 5n^3$
- $\boxed{\log_{10}(y)} = \log_{10} 5 + \log_{10} n^3 = \log_{10} 5 + 3\boxed{\log_{10}}\boxed{n}$

Definitely not a straight line.

- But what about this?
- So we use a log-log scale/plot

Polynomial on log-log plot

n	n^2	n^3	$20*n^3$
1	1	1	20
2	4	8	160
4	16	64	1280
8	64	512	10240
16	256	4096	81820
32	1024	32768	655360



slope: degree
shift: multiplicative factor

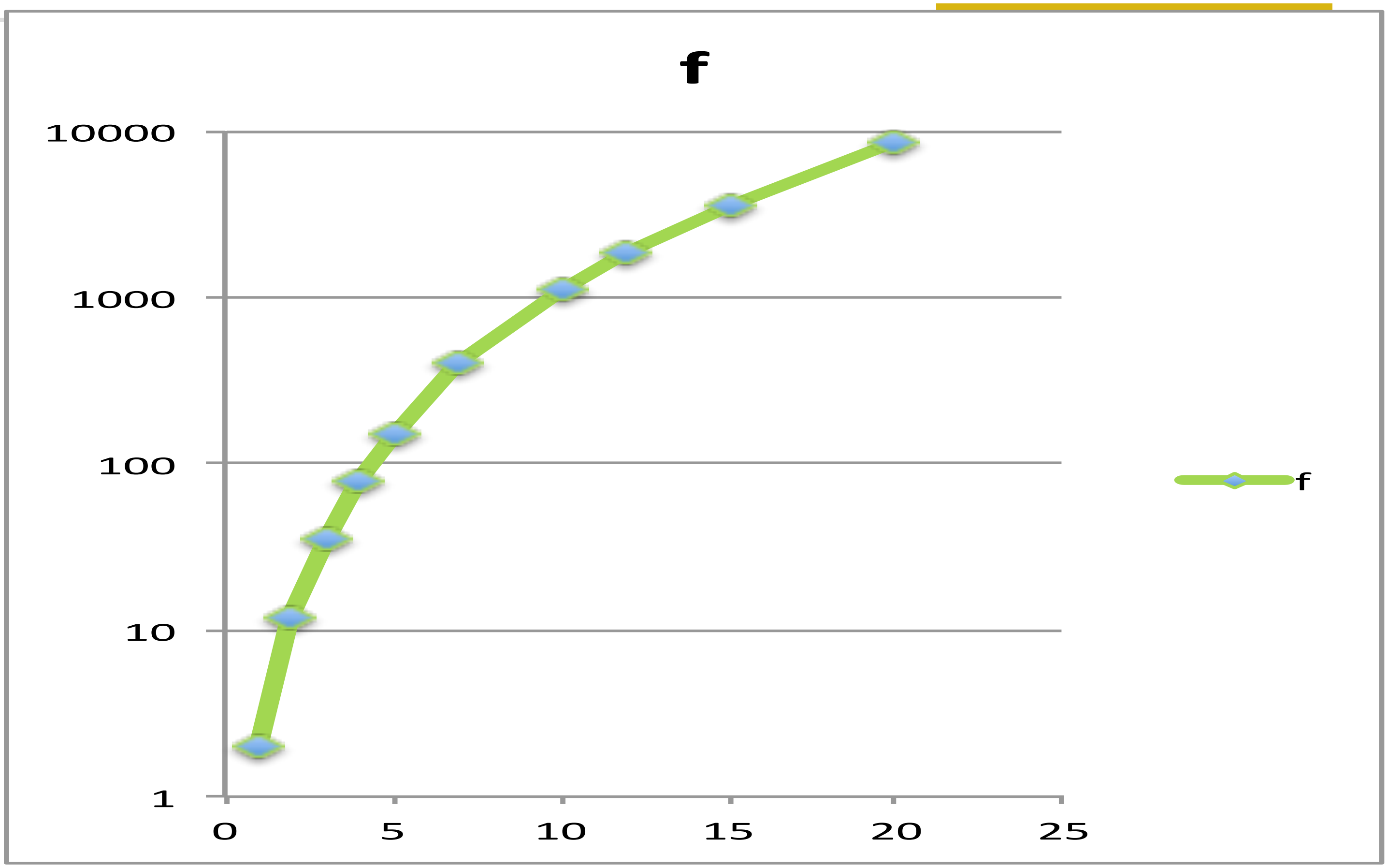
Handling multiple terms

- Functions like $f(n) = 3^n + 4^n$ and polynomials that have more than one term. We don't have a simple algebraic rule to compute logarithms of the sum of multiple terms
  - Now, $f(n) = 3^n + 4^n = 4^n\left(1 + \left(\frac{3}{4}\right)^n\right)$
    - and since $\left(\frac{3}{4}\right) < 1$, so $\lim_{n\to\infty}\left(1 + \left(\frac{3}{4}\right)^n\right) = 1$
  - so, as $n \to \infty$, we have $\log f(n) \to \log 4^n \times 1 = \log 4 \times n$
    i.e., only the dominant term matters
- For a polynomial like $f(n) = 4\times n^3 + 3\times n^2$ we do the same thing $f(n) = 4n^3\left(1 + \frac{3}{4n}\right)$ and
  as $n \to \infty$, the term in parentheses approaches 4,
  so $\log f(n) \to \log 4\times n^3 = \log 4 + 3\log n$

- *Message: when plotting your data, look for the trend among the points with larger input values*

Infer the function

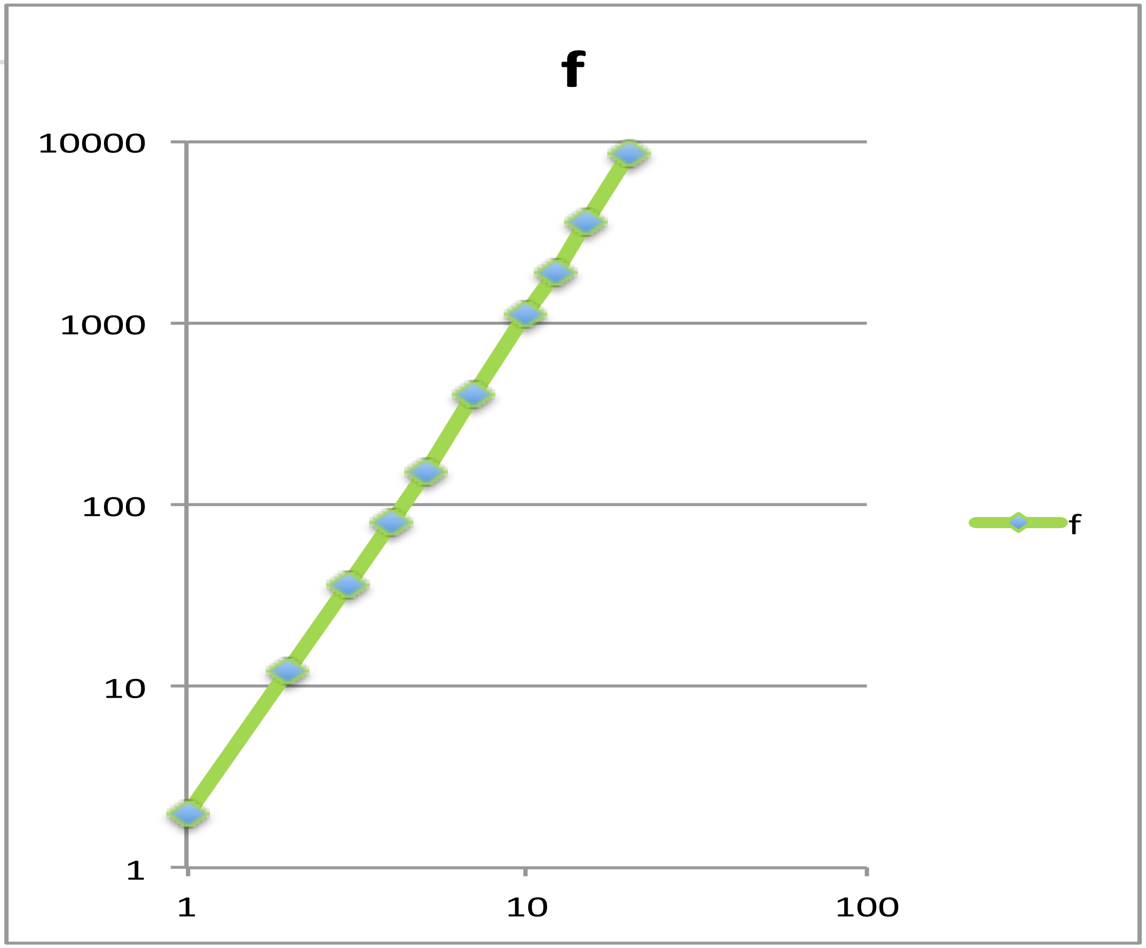
n	f(n)
1	2
2	12
3	36
4	80
5	150
7	400
10	1100
12	1872



The semi-log plot does not give a straight line, so f is not exponential

What about a log-log plot

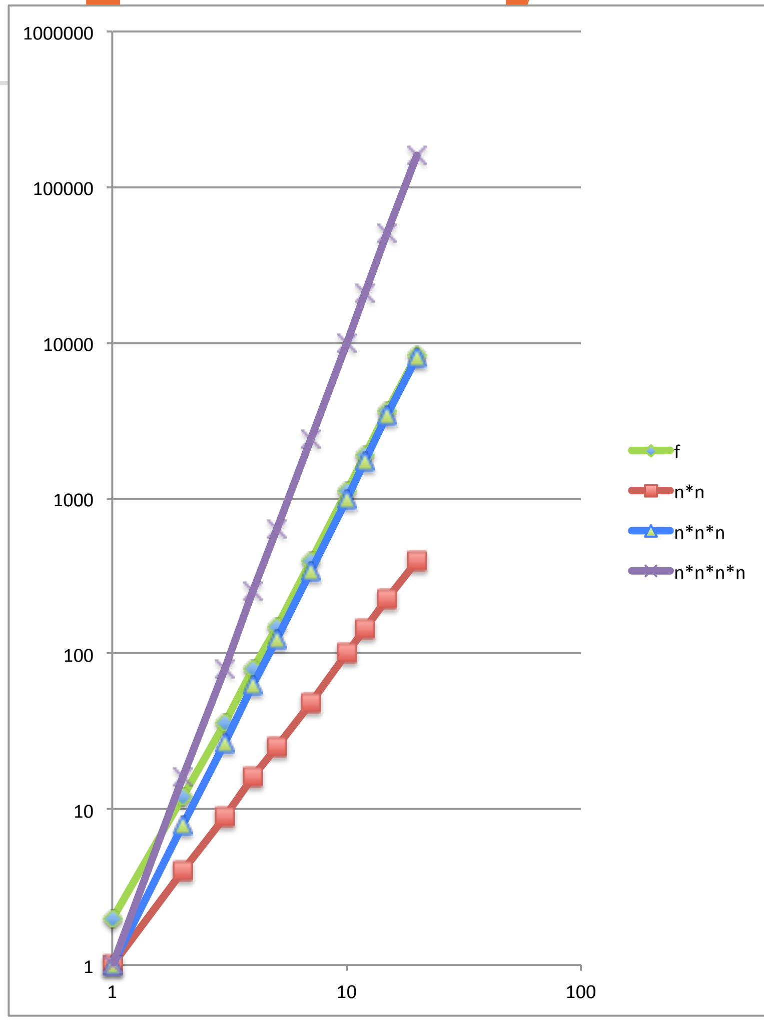
n	f(n)
1	2
2	12
3	36
4	80
5	150
7	400
10	1100
12	1872



YES! The log-log plot is asymptotically a straight line, so f is polynomial, but what is its leading term?

Continue empirically

n	f(n)	n^2	n^3	n^4
1	2	1	1	1
2	12	4	8	16
3	36	9	27	81
4	80	16	64	256
5	150	25	125	625
7	400	49	343	2401
10	1100	100	1000	10000
12	1872	144	1728	20736



Compare with n, n^2,n^3,n^4

It is degree 3, no multiplicative factor

Function Clubs

The Polynomial Club

- All *polynomial* functions are members:
  - $f(n)$ is in the club iff $f = \theta(n^k)$ for some constant, k.
- *Membership test*: to enter the club you scan your id
  - checker is just a *log-log plotter* you're in if it's a straight line with slope between 0° to 90°
- *Slowest* growing polynomial (*fastest* algorithms) are polynomials $f(n) = n^\varepsilon$ where, ϵ is an arbitrarily small constant.
- *Fastest* growing polynomial (*slowest* algorithms) $f(n) = n^\Gamma$ where, Γ is an arbitrarily large constant

The Exponential Club

- All *exponential* functions are members
- *Membership test*: to enter the club you scan your id
  - checker is just a *semi-log plotter* you're in if it's a straight line with slope between 0° to 90°
- *Slowest* growing exponential (*fastest* algorithms) are exponential $f(n) = \varepsilon^n$ where, ϵ is an arbitrarily small constant.
- *Fastest* growing exponential (*slowest* algorithms) $f(n) = \Gamma^n$ where, Γ is an arbitrarily large constant.

Are there other clubs?

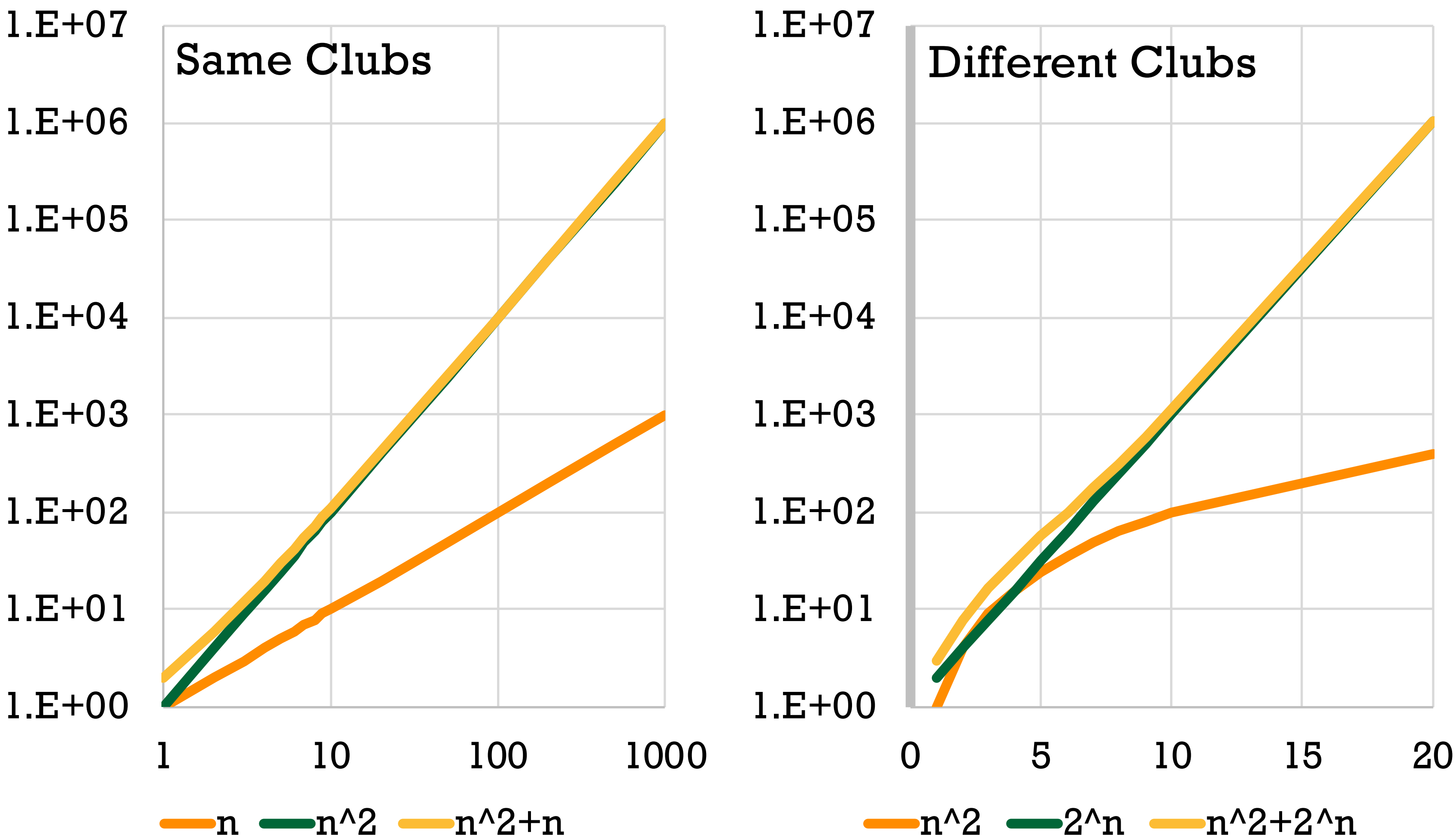
- The basic mathematical definition of $\lessdot$, $\gtrdot$, O and Ω still hold: for large enough n one function exceeds the other,
- The plotting trick is simply to compress the x or y axis plotting, and it doesn't change asymptotic behavior
- What if we compress the x axis and not the y axis: a so-called *log-semi* plot (but this naming convention is soon going to prove inadequate)
  - These are the *poly-log functions*: polynomials of $\log n$
  - The worst poly-log algorithm is faster the fastest polynomial algorithm $\log^{\Gamma} n \lessdot n^{\varepsilon}$
- *Super-exponential functions*: straight line when we plot $\log \log f(n)$ vs n

Additive Slowdown review

Universally, when adding functions, just ignore the slower growing function

- This carries over to membership tests
  - We saw this with multiple terms (slide 31) when they were members of the same club
  - If they are members of the same club, it's even more pronounced. $g(x) = f_1(x) + f_2(x)$ is a member of the faster growing club

Additive Slowdown



Scaling Variables

- Let $y = f(x)$ be an arbitrary (asymptotically monotonically increasing) function that represents the execution time of a program on an input of size x.

- We can massage/compress each of the variables x or y in two ways
  - $y' = \log y$ compress the dependent variable/vertically
  - $y'' = \log \log y$ double compress vertically
  - $x' = \log x$ compress independent variable/horizontally
  - $x'' = \log \log x$ double compress horizontally

Membership defining functions

Substituting the *scaling variables* yields the following nine *massaging functions* for the normal, log, and log log cases of each variable. This allows us to massage the input and output data for the different frames of reference in the graphs.

- $y = h_0(x)$ No massaging (normal plot, where we started)
- $y' = h_1(x)$ Compress the y axis (semi-log plot, *exponential*)
- $y'' = h_2(x)$ Double compress y (*doubly exponential* club)
- $y = h_3(x')$ Compress x (what does this do)?
- $y = h_4(x'')$ Double compress x (what does this do)?
- $y' = h_5(x')$ Compress both (*polynomial* club)
- $y' = h_6(x'')$ Compress y, double compress x (*polylog* club)
- $y'' = h_7(x')$ Double compress y, compress x
- $y'' = h_8(x'')$ Double compress both

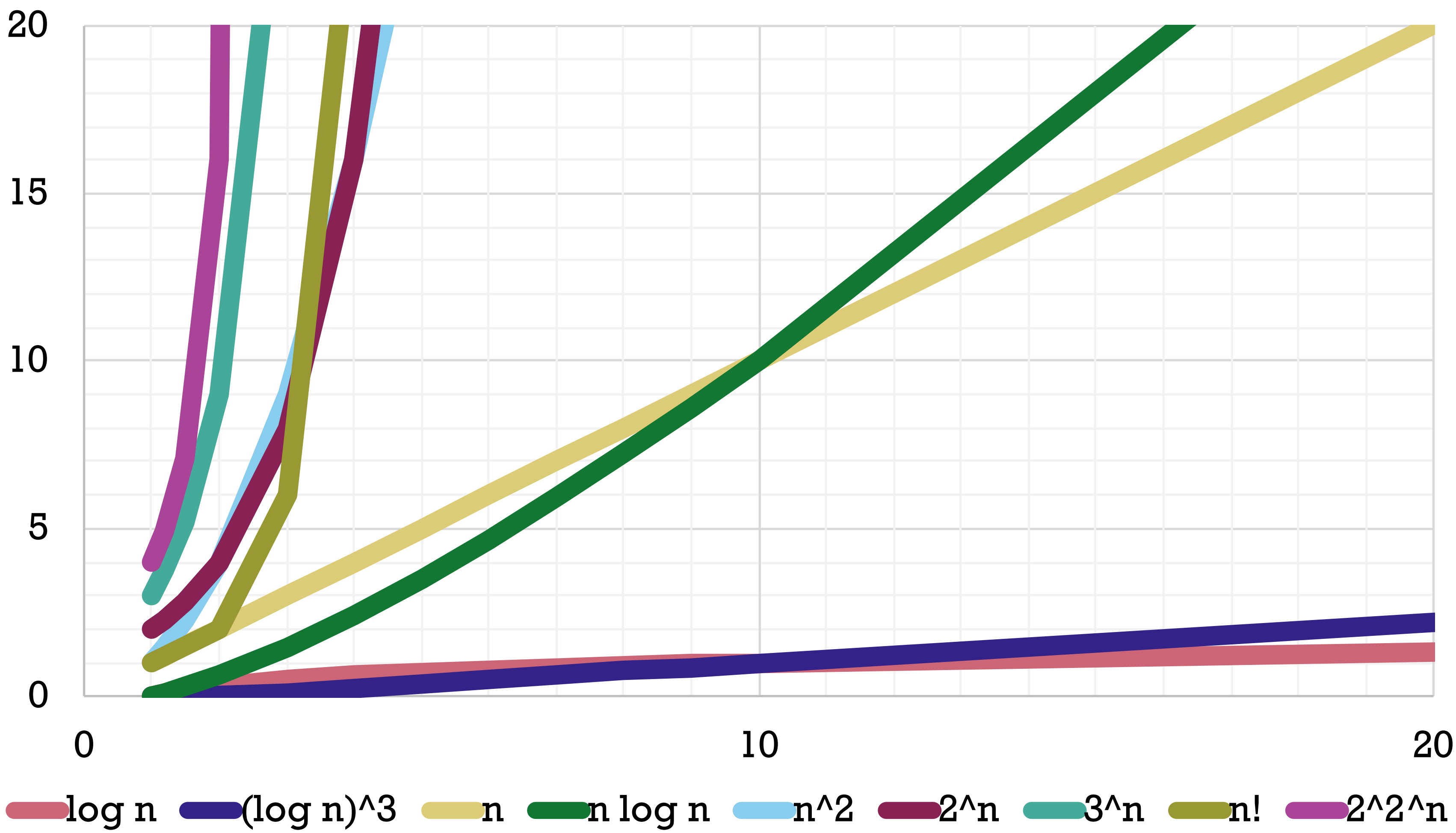
Swooping

Consider three clubs, C_a, C_b and C_c, with *membership defining functions*, h_a, h_b, and h_c, where C_a grows faster than C_b, and and C_b, grows faster than C_c. Membership defining functions are the massaging functions corresponding to these clubs.

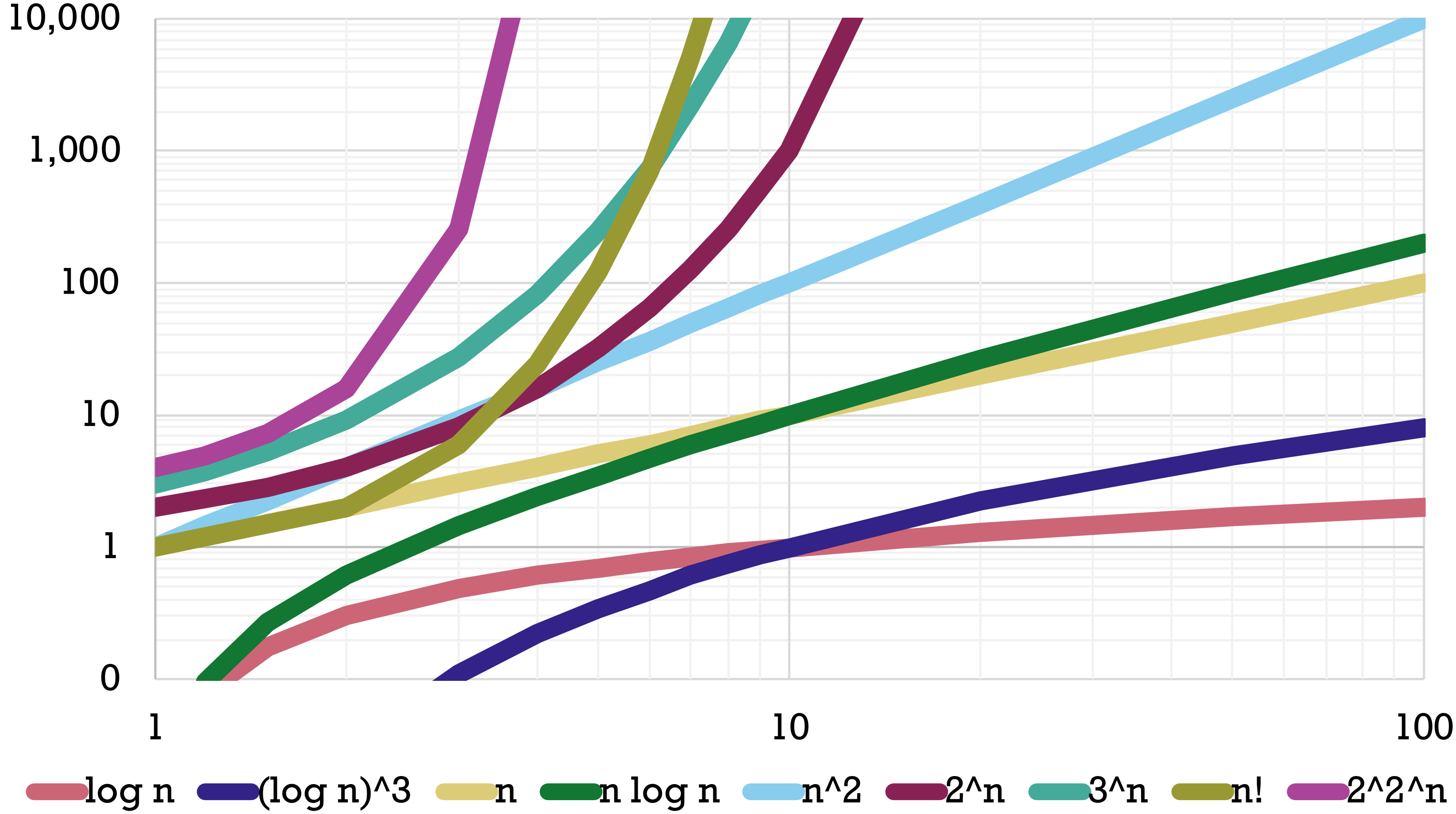
Consider a function $f_b(x) \in C_b$.

- $f_b(x)$ *swoops up* with respect to h_c: it grows *faster* than the any other function in C_c, i.e., even faster than a straight line with slope approaching 90°.
- $f_b(x)$ *swoops right* with respect to h_a: it grows *slower* than any other function in C_a, i.e., even slower than a straight line with slope approaching 0°.

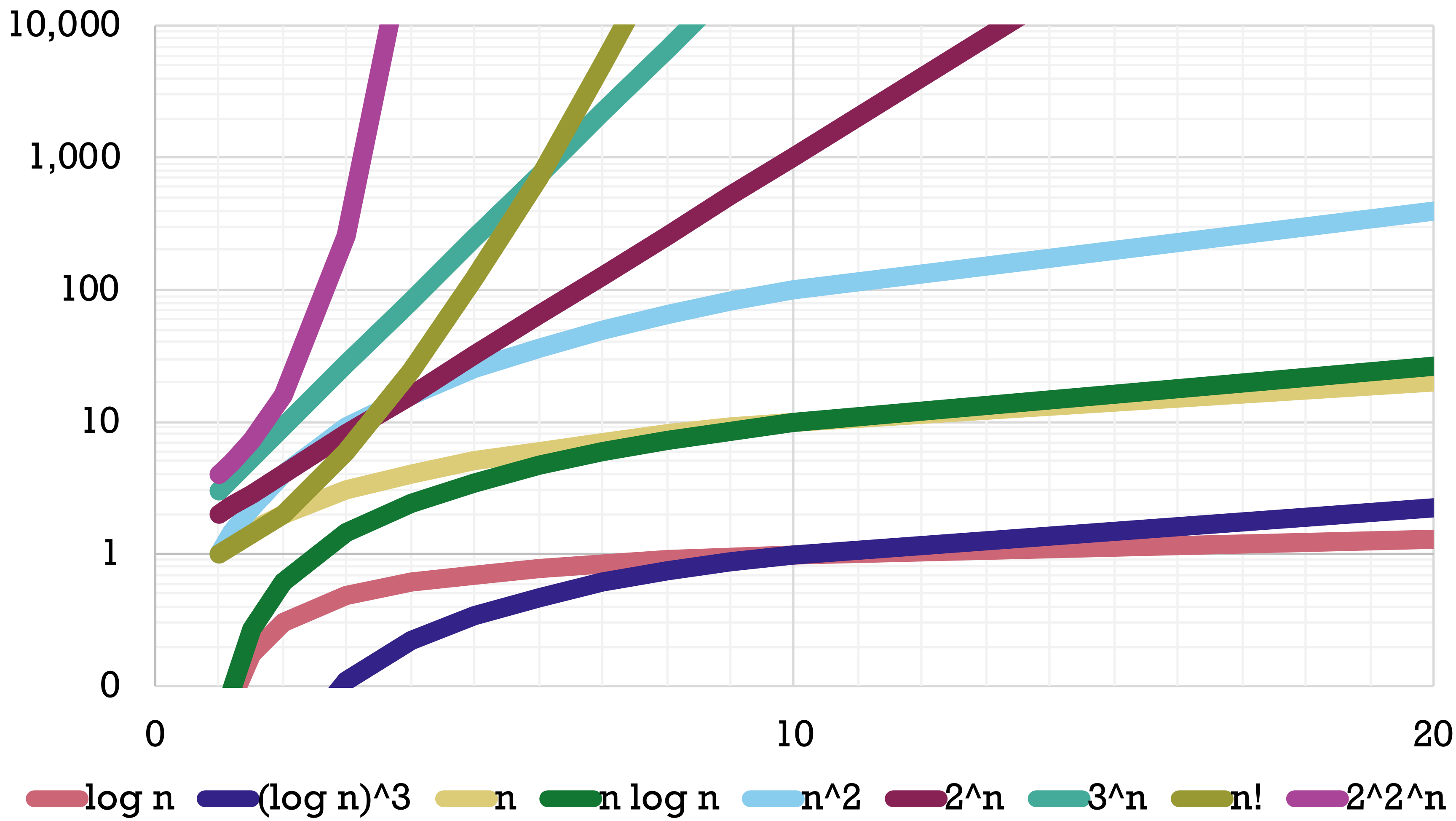
Normal: $y = h_0(x)$



Polynomial Club: $y' = h_5(x')$



Exponential Club: $y' = h_1(x)$



$y = h_3(x')$