Dynamic Programming

Cormen et. al. IV 15

Motivating Example: Fibonacci numbers

$F(1) = F(2) = 1$
$F(n) = F(n-1) + F(n-2) \quad n>2$

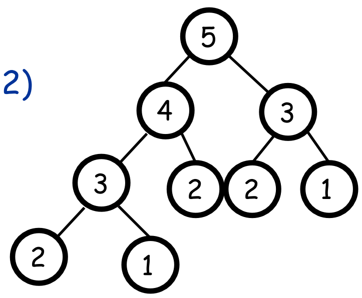
Fibonacci numbers

F(1) = F(2) = 1
F(n) = F(n-1) + F(n-2)   n>2

Simple recursive solution:

```
def fib(n) :
  if n<=2: return 1
  else: return fib(n-1) + fib(n-2)
```

What is the size of the call tree?



Fibonacci numbers

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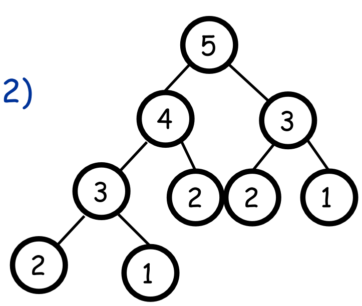
F(n) = F(n-1) + F(n-2)   n>2

Simple recursive solution:

```
def fib(n) :
  if n<=2: return 1
  else: return fib(n-1) + fib(n-2)
```

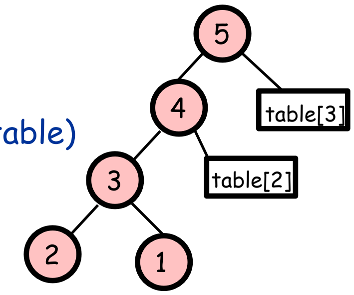
Problem: exponential call tree

Can we avoid it?



Efficient computation using a memo table

```
def fib(n, table) :
   # pre: n>0, table[i] either 0 or contains fib(i)
   if n <= 2 :
       return 1
   if table[n] > 0 :
       return table[n]
   result = fib(n-1, table) + fib(n-2, table)
   table[n] = result
   return result
```



We use a memo table, never computing the same value twice. How many calls now? O(n)

Can we do better?

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Look ma, no table

```
def fib(n) :
  if n<=2 : return 1
  a,b = 1
  c = 0
  for i in range(3, n+1) :
      c = a + b
      a = b
      b = c
   return c
```

Compute the values "bottom up"

Avoid the table, only store the previous two

same O(n) time complexity, constant space.

Only keeping the values we need.

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Optimization Problems

In optimization problems a set of **choices** are to be made to arrive at an optimum, and sub problems are encountered.

This often leads to a **recursive** definition of a solution. However, the recursive algorithm is often **inefficient** in that it solves the **same sub problem many times.**

Dynamic programming avoids this repetition by solving the problem **bottom up** and **storing** sub solutions, that are (still) needed.

Dynamic vs Greedy, Dynamic vs Div&Co

Compared to Greedy, there is **no predetermined local choice** of a sub solution, but a solution is chosen by computing a set of alternatives and **picking the best**.

Another way of saying this is: Greedy only needs ONE best solution.

Dynamic Programming **builds on** the recursive definition of a divide and conquer solution, but **avoids re-computation** by storing earlier computed values, thereby often saving orders of magnitude of time.

Fibonacci: from exponential to linear

Dynamic Programming

Dynamic Programming has the following steps

- Characterize the **structure** of the problem, i.e., show how a larger problem can be solved using solutions to sub-problems

- **Recursively** define the optimum

- Compute the optimum **bottom up**, **storing** values of sub solutions

- Construct the optimum from the **stored data**

Optimal substructure

Dynamic programming works when a problem has **optimal substructure**: we can construct the optimum of a larger problem from the optima of a "small set" of smaller problems.
- small: polynomial

Not all problems have optimal substructure. Travelling Salesman Problem (TSP) does not have optimal substructure.

Weighted Interval Scheduling

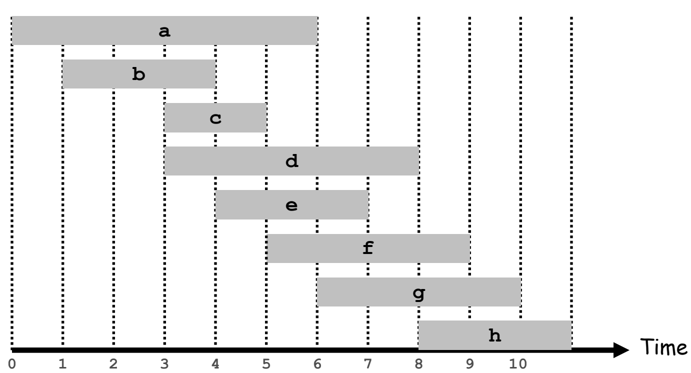
We studied a greedy solution for the interval scheduling problem, where we searched for the maximum number of compatible intervals.

If each interval has a weight and we search for the set of compatible intervals with the maximum sum of weights, no greedy solution is known.

Weighted Interval Scheduling

Weighted interval scheduling problem.

- Job j starts at s_j, finishes at f_j, and has value v_j .
- Two jobs compatible if they don't overlap.
- Goal: find maximum value subset of compatible jobs.



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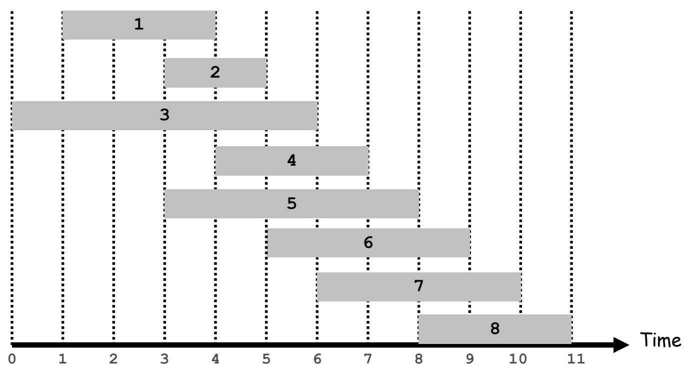
Weighted Interval Scheduling

Assume jobs sorted by finish time: $f_1 \le f_2 \le \ldots \le f_n$.
p(j) = largest index i < j such that job i is compatible with j,
in other words: p(j) is j's **latest predecessor**; p(j) = 0 if j has no predecessors. Example: p(8) = 5, p(7) = 3, p(2) = 0.
Using p(j) can you think of a recursive solution?



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Recursive (either / or) Solution

Notation. OPT(j): optimal value to the problem consisting of job requests 1, 2, ..., j.

- Case 1: OPT(j) includes job j.
  - add v_j to total value
  - can't use incompatible jobs { p(j) + 1, p(j) + 2, ..., j - 1 }
  - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., p(j)
- Case 2: OPT(j) does not include job j.
  - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., j-1

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max \left\{ v_j + OPT(p(j)),\ OPT(j-1) \right\} & \text{otherwise} \end{cases}$$

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Either / or recursion

This is very often a first recursive solution method:

- either some item is in and then there is some consequence
- or it is not, and then there is another consequence, e.g. knapsack, see later slides:

Here: for each job j
  either j is chosen
    - add v_j to the total value
    - consider p_j next

  or it is not
    - total value does not change
    - consider j-1 next

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Weighted Interval Scheduling: Recursive Solution

```
input: s_1,…,s_n , f_1,…,f_n , v_1,…,v_n

sort jobs by finish times so that f_1 ≤ f_2 ≤ ... ≤ f_n.

compute p(1), p(2), …, p(n)

Compute-Opt(j) {
   if (j == 0)
      return 0
   else
      return max(v_j + Compute-Opt(p(j)),Compute-Opt(j-1))
}
```

What is the size of the call tree here?
How can you make it big, e.g. exponential?

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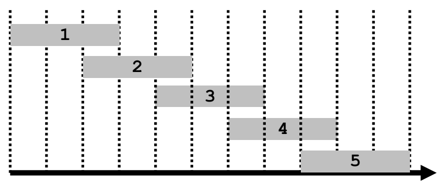
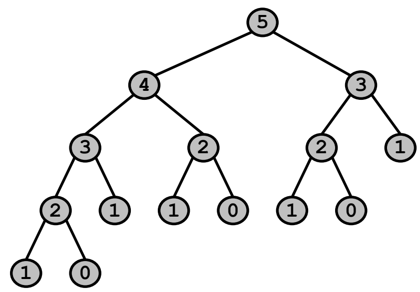
Analysis of the recursive solution

Observation. Recursive algorithm considers exponential number of (redundant) sub-problems.

Number of recursive calls for family of "layered" instances grows like Fibonacci sequence.





p(1) = 0, p(j) = j-2
Code on previous slide becomes Fibonacci: opt(j) calls opt(j-1) and opt(j-2)

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Weighted Interval Scheduling: Memoization

Memoization. Store results of each sub-problem in a cache; look up as needed.

```
input: n, s1,…,sn , f1,…,fn , v1,…,vn

sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.
compute p(1), p(2), …, p(n)

for j = 1 to n
   M[j] = empty
M[0] = 0            Global array

M-Compute-Opt(j) {
   if (M[j] is empty)
      M[j] = max(vj + M-Compute-Opt(p(j)),
                      M-Compute-Opt(j-1))
   return M[j]
}
```

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Weighted Interval Scheduling: Running Time

Claim. Memoized version of `M-Compute-Opt(n)` takes O(n log n) time.

- `M-Compute-Opt(n)` fills in all entries of M ONCE in constant time
- Since M has n+1 entries, this takes O(n)
- But we have sorted the jobs
- So Overall running time is O(n log n).

Weighted Interval Scheduling: Finding a Solution

Question. Dynamic programming computes optimal value. What if we want the choice vector determining which intervals are chosen.
Answer. Do some post-processing, walking BACK through the dynamic programming table.

```
Run Dynpro-Opt(n)
Run Find-Solution(n)

Find-Solution(j) {
   if (j = 0)
      output nothing
   else if (vj + M[p(j)] > M[j-1])
      print j
      Find-Solution(p(j))
   else
      Find-Solution(j-1)
}
```

Weighted Interval Scheduling: Bottom-Up

Bottom-up **dynamic programming**, **build a table.**

```
input: n, s_1,…,s_n , f_1,…,f_n , v_1,…,v_n

sort jobs by finish times so that f_1 ≤ f_2 ≤ ... ≤ f_n.

compute p(1), p(2), …, p(n)

Dynpro-Opt {
   M[0] = 0
   for j = 1 to n
      M[j] = max(v_j + M[p(j)], M[j-1])
}
```

By going in bottom up order M[p(j)] and M[j-1] are present when M[j] is computed. This takes O(nlogn) for sorting and O(n) for Compute, so O(nlogn)

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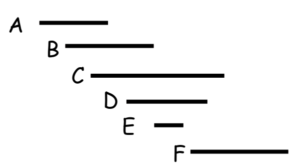
Do it, do it: Recursive

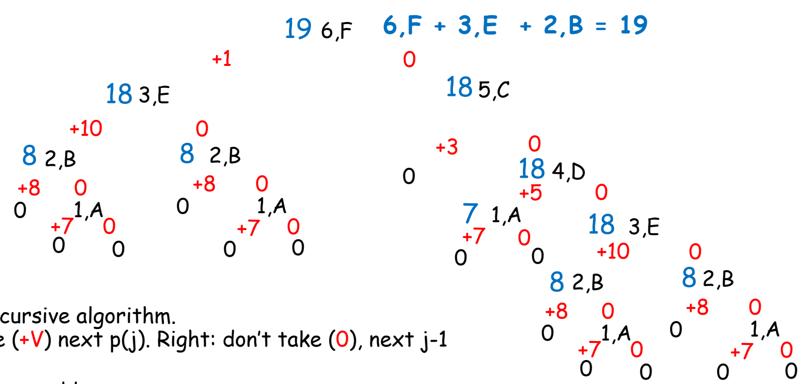
	S	F	V
A	1	5	7
B	2	9	8
C	4	13	3
D	6	12	5
E	9	10	10
F	11	15	1



Sort in F order
1 A
2 B
3 E
4 D
5 C
6 F

Determine p array
1,A: 0
2,B: 0
3,E: 2,B
4,D: 1,A
5,C: 0
6,F: 3,E

6,F + 3,E + 2,B = 19



Do the recursive algorithm.
Left: take (+V) next p(j). Right: don't take (0), next j-1

Up: edge: add,
node: take the max

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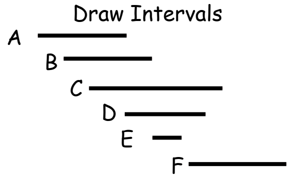
Do it, do it: Dynamic Programming

	S	F	V
A	1	5	7
B	2	9	8
C	4	13	3
D	6	12	5
E	9	10	10
F	11	15	1

Draw Intervals



Sort in F order

1 A
2 B
3 E
4 D
5 C
6 F

Determine p array

1,A: 0
2,B: 0
3,E: 2,B
4,D: 1,A
5,C: 0
6,F: 3,E

```
M[0] = 0
   for j = 1 to n
      M[j] = max(vj + M[p(j)],
                 M[j-1])
```

Create M table

0	7	8	18	18	18	19
0	1,A	2,B	3,E	4,D	5,C	6,F

Walk back to determine choices

6,F: take gets you 19, don't gets you 18, so take F
3,E: take gets you 18, don't gets you 8, so take E
2,B: take gets you 8, don't gets you 0, so take B

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Computing the p array

Claim. Memoized version of `M-Compute-Opt(n)` takes O(n log n) time.

- `M-Compute-Opt(n)` fills in all entries of M ONCE in constant time
- Since M has n+1 entries, this takes O(n)
- But we have sorted the jobs
- So Overall running time is O(n log n).

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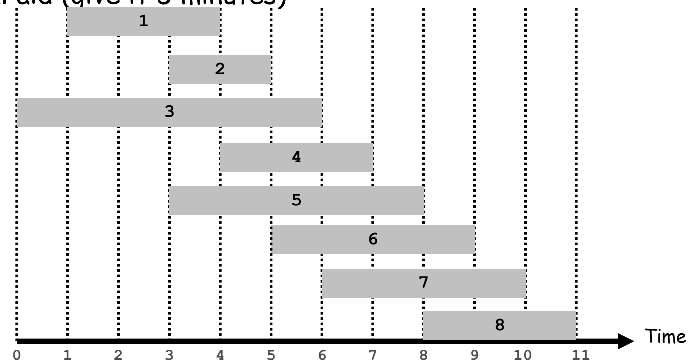
Computing the latest-predecessor array

Visually, it is "easy" to determine p(i), the largest index i < j such that job i is compatible with j. For the example below:

p[1…8] = [0, 0, 0, 1, 0, 2, 3, 5]

How about an algorithm? Or even as a human, try it without the visual aid (give it 5 minutes)



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How about an algorithm? Or even as a human, try it without the visual aid (give it 5 minutes)

Activity	A1	A2	A3	A4	A5	A6	A7	A8
Start (s)	1	3	0	4	3	5	6	8
Finish (f)	4	5	6	7	8	9	10	11
P								

Time

Computing the latest-predecessor array

Spoiler alert:

1. Treat all the start and finish times as "events" and sort them in increasing order (resolve ties any way, as long as all the f events are before the s events)
2. Have global variables LFSF and IFLSF (for "Latest_Finish_So_Far," and "Index_of_FLSF")
3. Process events in order as follows:
   a. If it is a finish event, f_i then update LFSF and ILFSF
   b. If it is a start event, s_i then set p(i) to ILFSF

Evnt	LFSF	ILFSF	p(x)=y
s3	0	0	p(3)=0
s1	0	0	p(1)=0
s2	0	0	p(2)=0
s5	0	0	p(5)=0
f1	4	1	
s4	4	1	p(4)=1
f2	5	2	
s6	5	2	p(6)=2
f3	6	3	
s7	6	3	p(7)=3
f4	7	4	
f5	8	5	
s8	8	5	p(8)=5
f6	9	6	
f7	10	7	
f8	11	8	

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Discrete Optimization Problems

Discrete Optimization Problem (S,f)

- S:
  - Set of solutions of a problem, satisfying some constraint
- $f : S \rightarrow R$
  - Cost function associated with feasible solutions
- Objective: find an optimal solution x_{opt} such that
  $f(x_{opt}) \leq f(x)$ for all x in S (minimization)
  or $f(x_{opt}) \geq f(x)$ for all x in S (maximization)

- Ubiquitous in many application domains
  - planning and scheduling
  - VLSI layout
  - pattern recognition
  - bio-informatics

Knapsack Problem

Knapsack problem.

- Given n objects and a "knapsack" of capacity W
- Item i has a weight w_i > 0 and value or profit v_i > 0.
- Goal: fill knapsack so as to maximize total value.

What would be a Greedy solution?

repeatedly add item with maximum v_i / w_i ratio …

Does Greedy work?

Capacity M = 7, Number of objects n = 3
w = [5, 4, 3]
v = [10, 7, 5] (ordered by v_i / w_i ratio)

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Either / or Recursion for Knapsack Problem

Notation: OPT(i, w) = optimal value of max weight subset that uses items 1, …, i with weight limit w.

- Case 1: item i is not included:
  - OPT includes best of { 1, 2, …, i-1 } using weight limit w
- Case 2: item i is included, if it can be included: w_i <= w
  - new weight limit = w – w_i
  - OPT includes best of { 1, 2, …, i–1 } using weight limit w-w_i

$$OPT(i,w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1, w) & \text{if } w_i > w \\ \max\{ OPT(i-1, w),\ \ v_i + OPT(i-1, w-w_i) \} & \text{otherwise} \end{cases}$$

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Knapsack Problem: Dynamic Programming

Knapsack. Fill an n+1 by W+1 array.

```
Input: n, W, weights w1,…,wn,
             values v1,…,vn

for w = 0 to W
 M[0, w] = 0

for i = 1 to n
 for w = 0 to W
  if wi > w :
     M[i, w] = M[i-1, w]
  else :
     M[i, w] = max (M[i-1, w],
                 vi + M[i-1, w-wi ])
return M[n, W]
```

Do it for:

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

W = 11

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Knapsack Algorithm

W + 1

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }												
{ 1, 2 }												
{ 1, 2, 3 }												
{ 1, 2, 3, 4 }												
{ 1, 2, 3, 4, 5 }												

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

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Knapsack Algorithm

W + 1

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }												
{ 1, 2, 3 }												
{ 1, 2, 3, 4 }												
{ 1, 2, 3, 4, 5 }												

At 1,1 we can fit item 1 and from then on, all we have is item 1

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

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Knapsack Algorithm

W + 1

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }												
{ 1, 2, 3, 4 }												
{ 1, 2, 3, 4, 5 }												

At 2,2 we can either not take item 2 (value 1 (previous row[2]) or we can take item 2 (value 6 previous row[0]+ 6)
At 2,3 we can either not take item 2 (value 1) or we can take item 2 and item 1 (value 7). From then on we can fit both items 1 and 2 (value 7)

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

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Knapsack Algorithm

W + 1 →

n + 1 ↓

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }												
{ 1, 2, 3, 4, 5 }												

From 3,0 to 3,4 we cannot take item 3.
At 3,5 we can either not take item 3 (value 7)
or we can take item 3 (value 18)
At 3,6 we can either not take item 3 (value 7)
or we can take item 2 (value 19), etc.,

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

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Knapsack Algorithm

W + 1 →

n + 1 ↓

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

OPT: 40

How do we find the objects in the optimum solution?

Walk back through the table!!

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

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Knapsack Algorithm

W + 1

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

OPT: 40
n=5 Don't take object 5 (7+28=35 < 40)

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

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Knapsack Algorithm

W + 1

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

OPT: 40
n=5 Don't take object 5
n=4 Take object 4 (18+22=40>25)

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

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Knapsack Algorithm

W + 1

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

OPT: 40
n=5 Don't take object 5
n=4 Take object 4
n=3 Take object 3

and now we cannot take anymore,
so choice set is {3,4},
choice vector is [0,0,1,1,0]

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

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Knapsack Problem: Running Time

Running time. $\Theta(nW)$.
- Not polynomial in input size!
  - W can be exponential in n
- Decision version of Knapsack is NP-complete. [Chapter 34 CLRS]

Knapsack approximation algorithm.
- There exists a poly-time algorithm that produces a feasible solution that has value within 0.01% of optimum.

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