Divide and Conquer: Counting Inversions

Rank Analysis

- Collaborative filtering
  - matches your preference (books, music, movies, restaurants) with that of others
  - finds people with similar tastes
  - recommends new things to you based on purchases of these people

- The basis of collaborative filtering:
  compare the **similarity of two rankings**

What's **similar**?

Given numbers 1 to n (the things) rank these according to your preference
- You get some permutation of 1..n
- Compare to someone else's permutation

Extreme similarity
- somebody else's ranking is exactly the same

Extreme dissimilarity
- somebody else's ranking is exactly the opposite

In the middle:
- count the **number of out of place rankings**

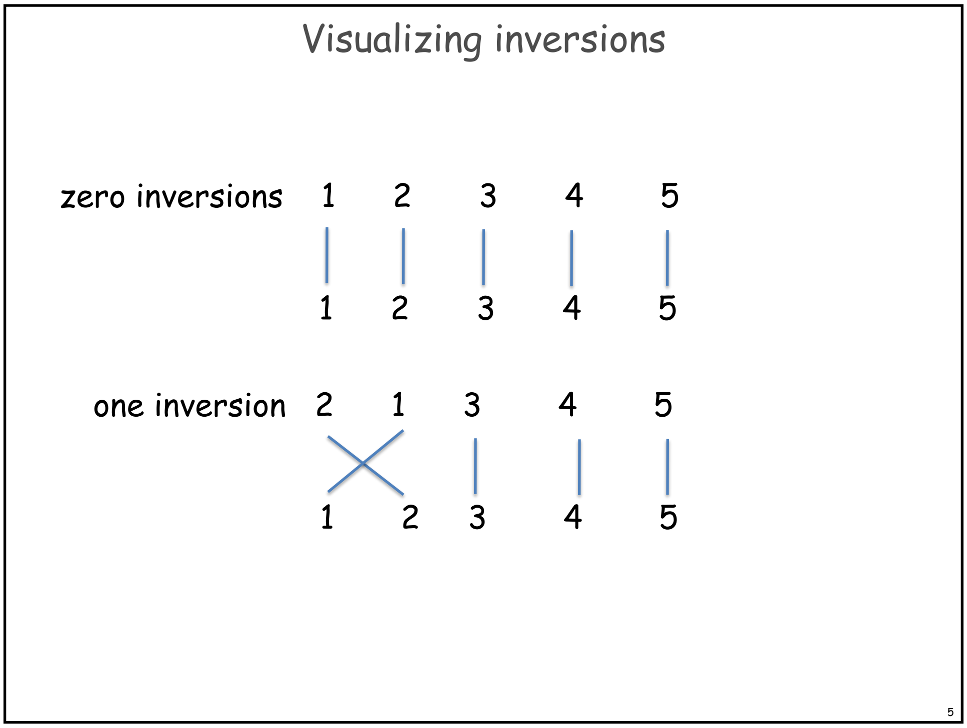
Simplify it

Count the number of **inversions** of a ranking
- $r_1, r_2, \ldots, r_n$
- count the number of out of order pairs
  - $i<j \quad r_i>r_j$
- eg: 2 1 4 3 5 2 inversions: (2,1) (4,3)

Why is this synonymous with comparing two different rankings?

Because we can re-number the things, such that one of the rankings (e.g. my ranking) becomes 1,2,...,n

my ranking: 1,2,...,5 your ranking 2,1,4,3,5
your #1 is my #2, your #2 is my #1
your #3 is my #4, your #4 is my #3


Visualizing inversions
zero inversions 1 2 3 4 5
1 2 3 4 5
one inversion 2 1 3 4 5
1 2 3 4 5
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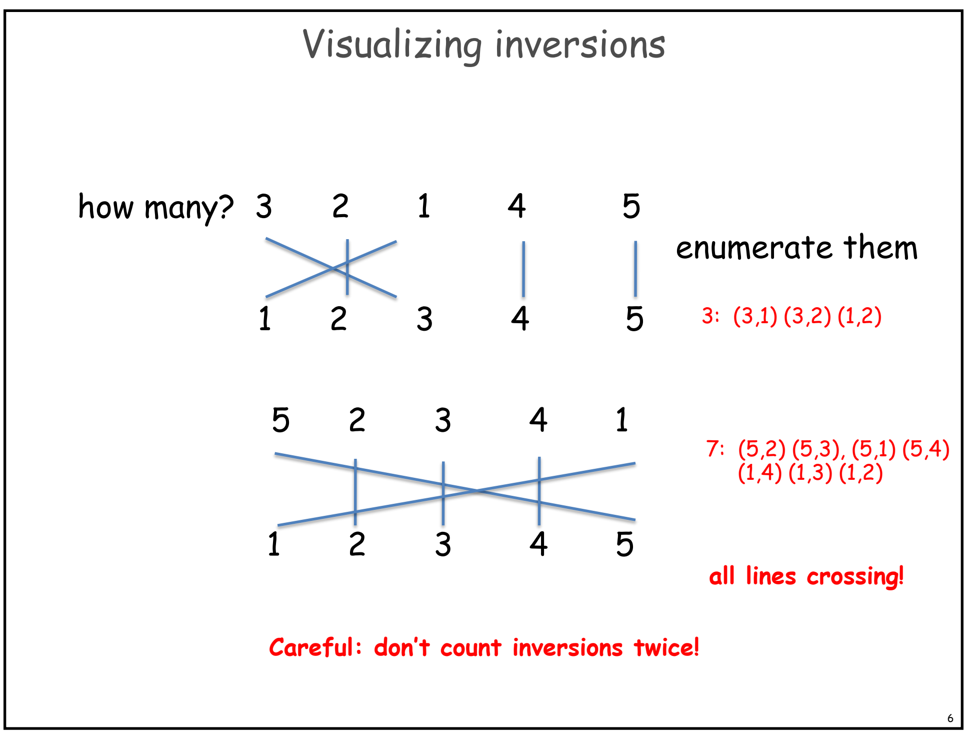
Visualizing inversions
how many? 3 2 1 4 5
1 2 3 4 5
enumerate them
3: (3,1) (3,2) (1,2)
5 2 3 4 1
1 2 3 4 5
7: (5,2) (5,3), (5,1) (5,4)
(1,4) (1,3) (1,2)
all lines crossing!
Careful: don't count inversions twice!
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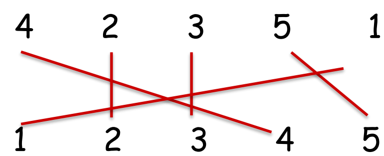
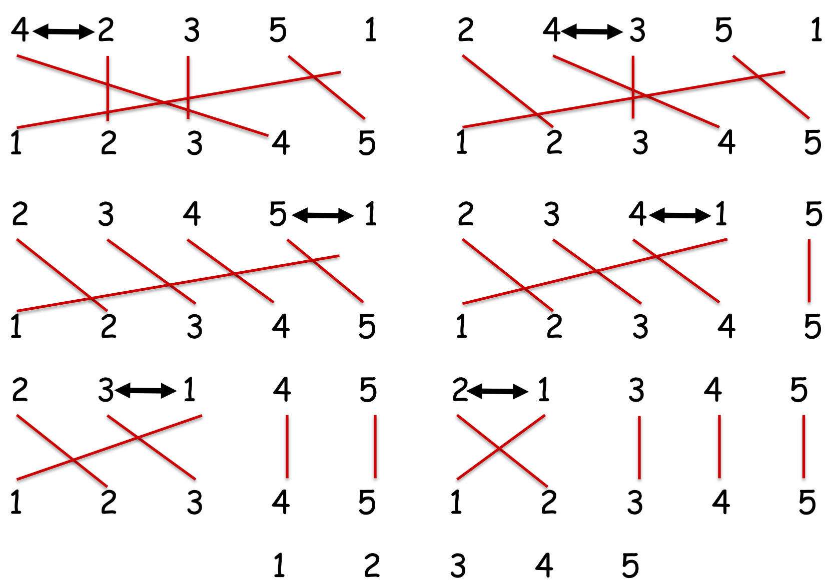
Sort

Does Bubble sort count inversions?
Bubble sort is $O(n^2)$

Do it on: 4 2 3 5 1 and see what happens



Do bubble sort, show each swap, count inversions



every swap takes out 1 inversion, and thus 1 line crossing

Can we do better?

Notice: there are potentially n*(n-1)/2 inversions. **WHY?**

Reverse order, all pairs are out of orders

Bubble sort counts each individual swap = inversion.To do better we must not count each individual inversion.

Think of merge sort

- in merge sort we do not swap consecutive elements that are out of order as in bubble sort, we make larger distance swaps
- if we can merge sort and keep track of the number of inversions we may get an O(n log n) algorithm
- Key observation: when an element from right is merged in, it "jumps" over all remaining elements of left !!

Eg: [4 2 3 5 1]

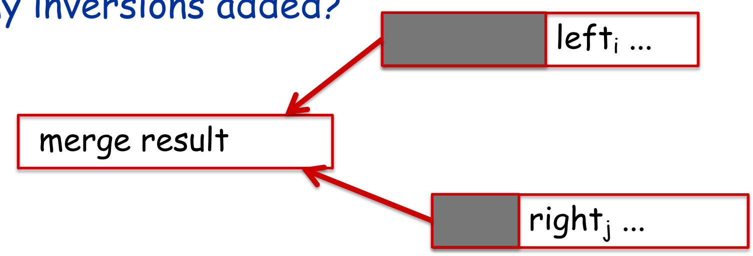
sort [4 2 3 5 1]

- sort LEFT: [4 2 3]
  - sort left: [4 2] → [2 4]:1 inversion
  - sort right: [3]
  - merge(left,right) → [2 3 4] 1 inversion (3 jumps over 4)
- sort RIGHT: [5 1] → [1 5] 1 inversion
- merge(LEFT,RIGHT) →[1 2 3 4 5]
  3 inversions (1 jumps over 2,3 & 4)

Total inversions: 1+1+1+3=6 (go check the visualization)

The algorithm

While merging in merge sort keep track of the number of inversions.
When merging an element from left: no inversions added
When merging an element from right: how many inversions added?



As many elements as are remaining in left,
because the element from the right jumps over all the remaining elements from left

Counting Inversions: Algorithm

```
count_inversions(list)
   if list has one element
      return 0
   divide list into two halves A and B
   rA = count_inversions(A)
   rB = count_inversions(B)
   rm = merge-and-count(A, B, list)
   return rA + rB + rm

merge-and-count(L, R, list)
  count = 0
  while L and R not empty:
    put smallest of Li and  Rj  in list
    if Rj smallest
        add number of elements remaining in L to count
  if L or R empty:
    append the other one to list
  return count
```

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Running time

Just like merge sort, the sort and count algorithm running time satisfies:

$$T(n) = 2\ T(n / 2) + cn$$

Running time is therefore $O(n \log n)$