1. Prove that $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$ for every positive integer n.

   Proof. We shall prove this using induction.
   In the basis step, $n = 1$, we see that
   $$1 = 1$$
   and
   $$\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = 1$$
   and so the basis step holds.
   In the inductive step, we will assume that $1+2+3+...+k = \frac{k(k+1)}{2}$ for some positive integer k and show that $1 + 2 + 3 + ... + k + (k + 1) = \frac{(k+1)(k+2)}{2}$. By the inductive hypothesis,
   $$1 + 2 + 3 + ... + k + (k + 1) = \frac{k(k+1)}{2} + (k + 1).$$
   With some algebraic manipulation this becomes
   $$1 + 2 + 3 + ... + k + (k + 1) = \frac{k(k+1) + 2(k+1)}{2}$$
   or
   $$1 + 2 + 3 + ... + k + (k + 1) = \frac{(k+1)(k+2)}{2}$$
   and so the inductive step holds.
   Since the inductive step and the basis step hold, it is true that $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$ for every positive integer n. □

2. Prove that $1 + 3 + 5 + ... + (2n - 1) = n^2$ for every positive integer n.

   Proof. We shall prove this using induction.
   In the basis step, $n = 1$, we see that
   $$2(1) - 1 = 1$$
   and
   $$1^2 = 1$$
   and so the basis step holds.
   In the inductive step, we will assume that $1 + 3 + 5 + ... + (2k - 1) = k^2$ for some positive integer k and show that $1+3+5+...+(2k-1)+(2(k+1)-1) = (k+1)^2$. By the inductive hypothesis,
   $$1 + 3 + 5 + ... + (2k - 1) + (2(k + 1) - 1) = k^2 + (2(k + 1) - 1) = k^2 + 2k + 1.$$
   Factoring this yields
   $$1 + 3 + 5 + ... + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2$$
   and so the inductive step holds.
   Since the inductive step and the basis step hold, it is true that $1+3+5+...+(2n-1) = n^2$ for every positive integer n. □

3. Prove that $2^n > n^2$ for every positive n that is greater than 4.

Proof. We shall prove this using induction.

In the basis step, $n = 5$, we see that

$$2^5 = 32 > 25 = 5^2$$

and so the basis step holds.

In the inductive step, we will assume $2^k > k^2$ for some positive integer k and show that $2^{k+1} > (k+1)^2$. Applying the inductive hypothesis,

$$2^{k+1} = 2 * 2^k > 2k^2 = k^2 + k^2.$$

Note that for $k \geq 3$,

$$k^2 - 2k - 1 = (k-1)^2 - 2 > 0,$$

so

$$k^2 > 2k + 1.$$

By substitution,

$$2^{k+1} > k^2 + k^2 > k^2 + 2k + 1 = (k+1)^2$$

and hence the inductive step holds.

Since the inductive step and the basis step hold, $2^n > n^2$ for every positive n that is greater than 4. $\square$

There are many different ways to show

$$k^2 > 2k + 1$$

for $k \geq 3$ - it may be useful to practice using induction to show it.

4. Prove that $n^5 - n$ is divisible by 5 for every positive integer n.

Proof. We shall prove this using induction.

In the basis step, $n = 1$,

$$n^5 - n = 1 - 1 = 0 = 5 * 0$$

so the basis step holds.

In the inductive step, we assume $k^5 - k$ is divisible by 5 for some positive integer k and we will show $(k+1)^5 - (k+1)$ is divisible by 5. Expanding the left-hand side yields,

$$(k+1)^5 - (k+1) = (k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1) - (k+1)$$

or, combining like terms except for $(k^5 - k)$,

$$(k+1)^5 - (k+1) = (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k).$$

Since this is the sum of two integers which are divisble by five, $(k+1)^5 - (k+1)$ is divisible by 5. Hence, the inductive step holds.

Since the inductive step and the basis step hold, $n^5 - n$ is divisible by 5 for every positive integer n. $\square$

5. Prove that $1 * 2 + 2 * 3 + 3 * 4 + ... + n * (n+1) = \frac{(n)(n+1)(n+2)}{3}$ for every positive integer n.

   Proof. We shall prove this using induction.

   In the basis step, $n = 1$,
   $$1 * 2 = 2$$
   and
   $$\frac{(1)(2)(3)}{3} = 2$$
   so the basis step holds.

   In the inductive step, we assume $1*2+2*3+3*4+...+k*(k+1) = \frac{(k)(k+1)(k+2)}{3}$ for some positive integer k and we will show $1*2+2*3+3*4+...+k*(k+1)+(k+1)*(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$. Applying the inductive hypothesis to the left-hand side yields
   $$1 * 2 + 2 * 3 + 3 * 4 + ... + k * (k+1) + (k+1) * (k+2) = \frac{(k)(k+1)(k+2)}{3} + (k+1) * (k+2)$$
   or
   $$1 * 2 + 2 * 3 + 3 * 4 + ... + k * (k+1) + (k+1) * (k+2) = \frac{(k)(k+1)(k+2)}{3} + \frac{3(k+1) * (k+2)}{3}$$
   which gives
   $$1 * 2 + 2 * 3 + 3 * 4 + ... + k * (k+1) + (k+1) * (k+2) = \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$$
   so
   $$1 * 2 + 2 * 3 + 3 * 4 + ... + k * (k+1) + (k+1) * (k+2) = \frac{(k+3)(k+1)(k+2)}{3}$$
   and hence the inductive step holds.

   Since the inductive step and the basis step hold, $1*2+2*3+3*4+...+n*(n+1) = \frac{(n)(n+1)(n+2)}{3}$ for every positive integer n. □

6. Find a formula for $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n}$ and prove it.

   The formula is $1 - 2^{-n}$. The proof is left to the reader.

7. Consider the sequence: $1 + 2 + 4 + 8 + 16 + ...$ What is the sum of the first n elements? Prove this.

   The sum is $2^{n+1} - 1$. The proof is left to the reader.