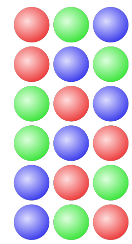
Permutations and Combinations

Rosen, Chapter 5.3

Motivating question

- In a family of 3, how many ways are there to arrange the members of the family in a line for a photograph?



A) 3 x 3
B) 3!
C) 3 x 3 x 3
D) 2^3

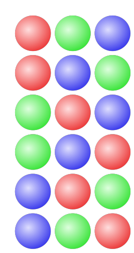
Permutations

- A permutation of a set of distinct objects is an ordered arrangement of these objects.
  - Example: (1, 3, 2, 4) is a permutation of the numbers 1, 2, 3, 4
- How many permutations of n objects are there?

How many permutations?

- How many permutations of n objects are there?
- Using the product rule:

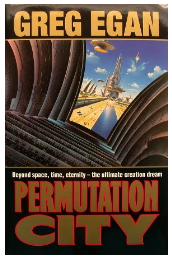
$$n \cdot (n - 1) \cdot (n - 2) ,\ldots, 2 \cdot 1 = n!$$



Anagrams

- Anagram: a word, phrase, or name formed by rearranging the letters of another.

Examples:
"cinema" is an anagram of iceman
"Tom Marvolo Riddle" =
"I am Lord Voldemort"



The anagram server: http://wordsmith.org/anagram/

Example

- Count the number of ways to arrange n men and n women in a line so that no two men are next to each other and no two women are next to each other.

a) n!
b) n! n!
c) 2 n! n!

Example

- You invite 6 people for a dinner party. How many ways are there to seat them around a round table? (Consider two seatings to be the same if everyone has the same left and right neighbors).

A) 6!
B) 5!
C) 7!

Example

- In how many ways can a photographer at a wedding arrange six people in a row, including the bride and groom, if
  - the bride must be next to the groom?
  - The bride is not next to the groom?
  - The bride is positioned somewhere to the left of the groom?

The Traveling Salesman Problem (TSP)

TSP: Given a list of cities and their pairwise distances, find a shortest possible tour that visits each city exactly once.

Objective: find a permutation $a_1,\ldots,a_n$ of the cities that minimizes

$$d(a_1, a_2) + d(a_2, a_3) + \ldots + d(a_{n-1}, a_n) + d(a_n, a_1)$$

where $d(i, j)$ is the distance between cities i and j

An optimal TSP tour through Germany's 15 largest cities

Solving TSP

- Go through all permutations of cities, and evaluate the sum-of-distances, keeping the optimal tour.
- Do we actually need to consider all permutations of n cities?

Generating Permutations

- Let's design a recursive algorithm for generating all permutations of {1,2,…,n}.

Generating Permutations

- Let's design a recursive algorithm for generating all permutations of {1,2,…,n}.
  - Starting point: decide which element to put first
  - what needs to be done next?
  - what is the base case?

Solving TSP

- Is our algorithm for TSP that considers all permutations of n-1 elements a feasible one for solving TSP problems with hundreds or thousands of cities?

r-permutations

- **r-permutation:** An ordered arrangement of r elements of a set.
- Example: List the 2-permutations of {a,b,c}.
  (a,b), (a,c), (b,a), (b,c), (c,a), (c,b)
- The number of r-permutations of a set of n elements:
  $P(n,r) = n(n-1)\dots(n-r+1) \quad (0 \le r \le n)$
  Example: $P(3,2) = 3 \times 2 = 6$
  Can be expressed as:
  $$P(n, r) = n! / (n-r)!$$
  Note that $P(n, 0) = 1$.

r-permutations - example

- How many ways are there to select a first-prize winner, a second prize winner and a third prize winner from 100 people who have entered a contest?

Question

- How many poker hands (five cards) can be dealt from a deck of 52 cards?
- How is this different than r-permutations?

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- How is this different than r-permutations?

In an r-permutation we cared about order. In this case we don't

Combinations

- An r-combination of a set is a subset of size r
- The number of r-combinations out of a set with n elements is denoted as C(n,r) or $\binom{n}{r}$
  - {1,3,4} is a 3-combination of {1,2,3,4}
  - How many 2-combinations of {a,b,c,d}?

Unordered versus ordered selections

- Two ordered selections are the same if
  - the elements chosen are the same;
  - the elements chosen are in the same order.
- Ordered selections**: r-permutations**.
- Two unordered selections are the same if
  - the elements chosen are the same.
    (regardless of the order in which the elements are chosen)
- Unordered selections: **r-combinations.**

Permutations or combinations?

- Determine if the situation represents a permutation or a combination:
  - In how many ways can three student-council members be elected from five candidates?
  - In how many ways can three student-council members be elected from five candidates to fill the positions of president, vice-president and treasurer
  - A DJ will play three songs out of 10 requests.

A) Permutations B) Combinations

Relationship between P(n,r) and C(n,r)

- Suppose we want to compute P(n,r) .
- Constructing an r-permutation from a set of n elements can be thought as a 2-step process:
  - Step 1: Choose a subset of r elements;
  - Step 2: Choose an ordering of the r-element subset.
- Step 1 can be done in C(n,r) different ways.
- Step 2 can be done in r! different ways.
- Based on the multiplication rule, P(n,r) = C(n,r) · r!
- Thus

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r! \cdot (n-r)!}$$

r-combinations

- How many r-combinations?

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Note that C(n, 0) = 1

Example: How many poker hands (five cards) can be dealt from a deck of 52 cards?

C(52,5) = 52! / (5!47!)

r-combinations

- How many r-combinations?

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Note that C(n, 0) = 1

- C(n,r) satisfies:

$$C(n,r) = C(n, n-r)$$

  - We can see that easily without using the formula

Combinations or permutations?

- How many bit strings of length n contain exactly r ones? P(n,r) or C(n,r)?

Example

- The faculty in biology and computer science want to develop a program in computational biology. A committee of 4 composed of two biologists and two computer scientists is tasked with doing this. How many such committees can be assembled out of 20 CS faculty and 30 biology faculty?

Example

- A coin is flipped 10 times, producing either heads or tails. How many possible outcomes
  - are there in total?
  - contain exactly two heads?
  - contain at least three heads?
  - contain the same number of heads and tails?

Example

- How many permutations of {a,b,c,d,e,f,g} end with a?

A) 5!
B) 6!
C) 7!
D) 6 x 6!

Example

- How many permutations of the letters ABCDEFGH contain the string ABC?

Example

- How many 10 character (digits and lowercase/uppercase letters) passwords are possible if

a) characters cannot be repeated?

b) characters can be repeated?

Some Advice about Counting

- Apply the multiplication rule if
  - The elements to be counted can be obtained through a multistep selection process.
  - Each step is performed in a fixed number of ways regardless of how preceding steps were performed.
- Apply the addition rule if
  - The set of elements to be counted can be broken up into disjoint subsets
- Apply the inclusion/exclusion rule if
  - It is simple to over-count and then to subtract duplicates

Some more advice about Counting

- Make sure that

  1) every element is counted;

  2) no element is counted more than once. (avoid double counting)

- When using the addition rule:

  1) every outcome should be in some subset;

  2) the subsets should be disjoint; if they are not, subtract the overlaps

Example using Inclusion/Exclusion Rule

How many integers from 1 through 100 are multiples of 4 or multiples of 7 ?

A: integers from 1 through 100 which are multiples of 4;

B: integers from 1 through 100 which are multiples of 7.

we want to find $|A \cup B|$.

$|A \cup B| = |A| + |B| - |A \cap B|$ (incl./excl. rule)

$A \cap B$ is the set of integers from 1 through 100 which are multiples of 28.

Computing C(n, k) recursively

- consider the nth object

C(n,k) = C(n-1,k-1) + C(n-1,k)
pick n or don't

C(n, k): base case

- C(k, k) = 1
  Why?
- C(n, 0) = 1
  Why?

Computing C(n, k) recursively

C(n,k) = C(n-1,k-1) + C(n-1,k)
pick n or don't
C(k,k) = 1
C(n,0) = 1

we can easily code this as a recursive method!

- This is an example of a **recurrence relation**, which is a recursive mathematical expression