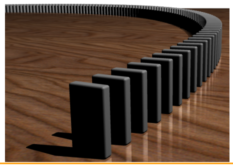
Mathematical Induction

Rosen Chapter 5



Motivation

- Show that any postage of ≥ 8¢ can be obtained using 3¢ and 5¢ stamps.
- First check for a few values:

  8¢ = 3¢ + 5¢

  9¢ = 3¢ + 3¢ + 3¢

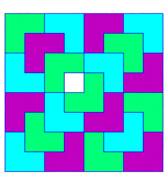
  10¢ = 5¢ + 5¢

  11¢ = 5¢ + 3¢ + 3¢

  12¢ = 3¢ + 3¢ + 3¢ + 3¢
- How to generalize this?

Motivation

- Let n be a positive integer. Show that every $2^n \times 2^n$ chessboard with one square removed can be tiled using triominoes, each covering three squares at a time.



Motivation

- Prove that for every positive value of n,

  $1 + 2 + ,\ldots, + n = n(n + 1)/2.$

Motivation

- Many mathematical statements have the form:

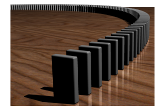
  $\forall n \in \mathbf{N}, P(n)$ P(n): Logical predicate
- Example: For every positive value of n,

  $1 + 2 + ,\ldots, + n = n(n + 1)/2.$
- Predicate – propositional function that depends on a variable, and has a truth value once the variable is assigned a value.
- Mathematical induction is a proof technique for proving such statements

Proving P(3)

- Suppose we know: P(1) and P(n) $\rightarrow$ P(n + 1) $\forall n \geq 1$.

  Prove: P(3)



- Proof:
  1. P(1). [premise]
  2. P(1) $\rightarrow$ P(2). [specialization of premise]
  3. P(2). [step 1, 2, & modus ponens]
  4. P(2) $\rightarrow$ P(3). [specialization of premise]
  5. P(3). [step 3, 4, & modus ponens]

  We can construct a proof for every finite value of n
- Modus ponens: if p and p $\rightarrow$ q then q

Example: $1 + 2 + \ldots + n = n(n + 1)/2.$

- $P(n)$: $1+2+\ldots+n = n(n + 1) / 2$
- Prove: $P(n{=}1)$: $1 =?\ 1(1 + 1)/2 = 1.$
- Assume: $P(n{=}k)$ *is true*

  $1+2+\ldots+k= k(k + 1)/2$
  - This is called the *Inductive Hypothesis*
- Use that assumption to prove $P(n{=}k{+}1)$ *is true.*

  $1+2+\ldots+k+k+1 =?\ (k + 1)(k + 2)/2.$

  $(1+2+..+k) + k+1 =?\ (k+1)(k+2)/2$

  $k(k+1)/2 + k+1 =?\ (k+1)(k+2)/2$ [by Ind. Hyp.]

  $k(k+1)/2 + 2(k+1)/2 =?\ (k+1)(k+2)/2$

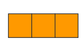
  $(k^2+k+2k+2)/2 =?\ (k+1)(k+2)/2$

  $(k^2+k+2k+2)/2 = (k^2 + k + 2k + 2)/2$

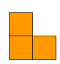
A Geometrical interpretation

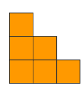
1: 

2: 

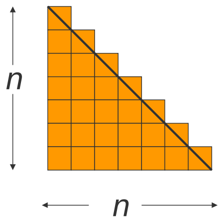
3: 

Put these blocks, which represent numbers, together to form sums:

1 + 2 = 

1 + 2 + 3 = 

A Geometrical interpretation



Area is $n^2/2 + n/2 = n(n + 1)/2$

The Principle of Mathematical Induction

- Let P(n) be a statement that, for each natural number n, is either true or false.
- To prove that $\forall n \in \mathbf{N}, P(n)$, it suffices to prove:
  - P(n=1) is true. (or sometimes some other n) (**base** case)
  - $\forall n \in \mathbf{N}, P(n) \rightarrow P(n + 1)$. (inductive **step**)
- This is not magic.
- It is a recipe for constructing a proof for an arbitrary $n \in \mathbf{N}$.

Mathematical Induction and the Domino Principle

If

the 1st domino falls over

and

the *n*th domino falls over implies that domino (n + 1) falls over

then

domino n falls over for all $n \in$ **N.**

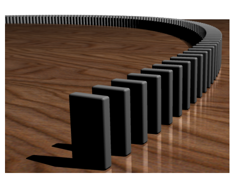


image from http://en.wikipedia.org/wiki/File:Dominoeffect.png

Proof by induction

- 3 steps:
  - Prove P(n=1). [the basis]
  - Assume P(n=k) [the induction hypothesis]
  - Using P(n=k), prove P(n=k+1) [the inductive step]

Example

- Show that any postage of ≥ 8¢ can be obtained using 3¢ and 5¢ stamps.
- First check for a few values:

  8¢ = 3¢ + 5¢

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- How to generalize this?

Example

- Let P(n) be the statement "n cents postage can be obtained using 3¢ and 5¢ stamps".
- Want to show that

  "P(n=k) is true" *implies* "P(n=k+1) is true"

  for all k ≥ 8.
- How can you increase k by 1?

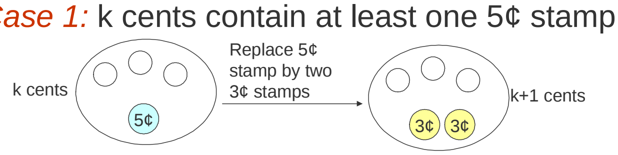
  2 cases:

  1) P(k) is true and the k cents contain at least one 5¢.

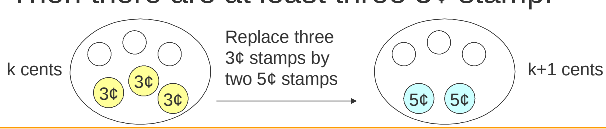
  2) P(k) is true and the k cents do not contain any 5¢.

Example

Case 1: k cents contain at least one 5¢ stamp.



Case 2: k cents do not contain any 5¢ stamp. Then there are at least three 3¢ stamp.



Examples

- Show that $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$
- Show that for $n \geq 4$ $2^n < n!$
- Show that $n^3 - n$ is divisible by 3 for every positive n.
- Show that $1 + 3 + 5 + \ldots + (2n+1) = (n+1)^2$
- Prove that a set with n elements has 2^n subsets

All horses have the same color

- *Base case:* If there is only one horse, there is only one color.
- *Induction step:* Assume as induction hypothesis that within any set of n horses, there is only one color. Now look at any set of n + 1 horses. Number them: 1, 2, 3, ..., n, n + 1. Consider the sets {1, 2, 3, ..., n} and {2, 3, 4, ..., n + 1}. Each is a set of only n horses, therefore within each there is only one color. But the two sets overlap, so there must be only one color among all n + 1 horses.

- This is clearly wrong, but can you find the flaw?

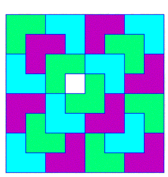
All horses have the same color

- The inductive step requires that k >= 2, otherwise there is no intersection! So P(2) should be the base case, which is obviously incorrect.

- In the book there is a similar example.

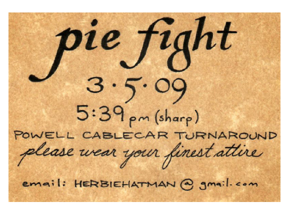
More induction examples

- Let n be a positive integer. Show that every $2^n \times 2^n$ chessboard with one square removed can be tiled using triominoes, each covering three squares at a time.



Odd Pie Fights

- An odd number of people stand at mutually distinct distances. Each person throws a pie at their nearest neighbor. Show that there is at least one survivor.



http://laughingsquid.com/san-francisco-pie-fight-at-the-powell-street-cable-car-turnaround/

Strong induction

- Induction:
  - P(n=1) is true.
  - $\forall k \in N$, P(n=k) $\rightarrow$ P(n=k+1).
  - Implies $\forall k \in N$, P(n=k)
- Strong induction:
  - P(n=1) is true.
  - $\forall k \in N$, (P(n=1) $\wedge$ P(n=2) $\wedge \ldots \wedge$ P(n=k)) $\rightarrow$ P(n=k+1)
  - Implies $\forall k \in N$, P(n=k) is true.

Example

- Prove that all natural numbers ≥ 2 can be represented as a product of primes.
- Basis: 2: 2 is a prime.
- Assume that 1, 2,…, n can be represented as a product of primes.

Example

- Show that n+1 can be represented as a product of primes.
  - If n+1 is a prime: It can be represented as a product of 1 prime, itself.
  - If n+1 is composite: Then, n + 1 = ab, for some a,b < n + 1.

    Therefore, $a = p_1p_2 \ldots p_k$ & $b = q_1q_2 \ldots q_l$, where all p_i and q_i are primes.

    Represent $n+1 = p_1p_2 \ldots p_kq_1q_2 \ldots q_l$ which is a product of primes!

Breaking chocolate

Theorem: Breaking up a chocolate bar with n “squares” takes n-1 breaks.

Induction and Recursion

- Induction is useful for proving correctness of recursive algorithms
- Example

```
// Returns base ^ exponent.
// Precondition: exponent >= 0
public static int pow(int x, int n) {
    if (n == 0) {
        // base case; any number to 0th power is 1
        return 1;
    } else {
        // recursive case:  x^n = x * x^(n-1)
        return x * pow(x, n-1);
    }
}
```

Induction and Recursion

```
public static int pow(int x, int n) {
    if (n == 0){
       return 1;
    } else {
       return x * pow(x, n-1);
    }
}
```

Claim: the algorithm correctly computes x^n.

Proof: By induction on n

Base case: n = 0: it correctly returns 1

Inductive step: assume that for n the algorithm correctly returns x^n.

Then for n+1 it returns $x \, x^n = x^{n+1}$.

Induction and Recursion

- n! of some integer n can be characterized as:
  n! = 1 for n = 0; otherwise
  n! = n (n - 1) (n - 2) ... 1
- Want to write a recursive method for computing it. We notice that n! = n (n – 1)!
- This is all we need to put together the method:

```
public static int factorial(int n) {
    if (n == 0) {
        return 1;
    } else {
        return n * factorial(n-1);
    }
}
```

Induction in CS

- Induction is a powerful tool for showing algorithm correctness – not just for recursive algorithms (CS320)