1. $P(n)$ $\quad 1+2+3+\cdots+n = n(n+1)/2 \quad n>0$

Basis $\quad P(n=1)$: $\quad 1 = 1(1+1)/2 = 2/2 = 1 \quad$ True

Ind. Hyp. $\quad P(n=k)$ assumed True. $\quad 1+2+\cdots+k = k(k+1)/2$

Ind. Step $\quad$ Prove $P(n=k) \rightarrow P(n=k+1)$

$P(n=k+1)$ definition. $\quad \underbrace{1+2+\cdots+k}+k+1 \stackrel{?}{=} (k+1)(k+2)/2$

by Ind. Hyp. $\quad k(k+1)/2 + k+1 \stackrel{?}{=} (k+1)(k+2)/2$

multiply by 2 $\quad k(k+1) + 2(k+1) \stackrel{?}{=} (k+1)(k+2)$

$(k+1)(k+2) = (k+1)(k+2)$

True

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2. $P(n)$ $\quad 1+3+5+\cdots+(2n-1) = n^2 \quad n>0$

Basis $\quad P(n=1)$ $\quad 1 \stackrel{?}{=} 1^2 = 1 \quad$ True

Ind. Hyp. $\quad P(n=k)$ assumed True $\quad 1+3+\cdots(2k-1) = k^2$

Ind. Step. $\quad$ Prove $P(n=k) \rightarrow P(n=k+1)$

def. of $P(n=k+1)$: $\quad \underbrace{1+3+\cdots+(2k-1)}+(2(k+1)-1) \stackrel{?}{=} (k+1)^2$

by Ind. Hyp. : $\quad k^2 + 2(k+1) - 1 \stackrel{?}{=} (k+1)^2$

$k^2+2k+1 \stackrel{?}{=} k^2+2k+1$

Yes - True

3. Prove $P(n)$: $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = n(n+1)(n+2)/3$ $\quad n > 0$

Basis. $P(n=1)$ $\quad 1\cdot 2 \stackrel{?}{=} 1(1+1)(1+2)/3 = 2\cdot 3/3 = 2$ True

Ind. Hyp: $P(n=k)$ assume True $\quad 1\cdot 2 + 2\cdot 3 + \cdots + k(k+1) = k(k+1)(k+2)/3$

Ind. Step. Prove $P(n=k) \rightarrow P(n=k+1)$

def. of $P(n=k+1)$ $\quad \underbrace{1\cdot 2 + 2\cdot 3 + \cdots + k(k+1)} + (k+1)(k+2) \stackrel{?}{=} (k+1)(k+2)(k+3)/3$

by Ind. Hyp.

$$k(k+1)(k+2)/3 + (k+1)(k+2) \stackrel{?}{=} (k+1)(k+2)(k+3)/3$$

multiply by 3 $\quad k(k+1)(k+2) + 3(k+1)(k+2) \stackrel{?}{=} (k+1)(k+2)(k+3)$

divide by $(k+1)(k+2)$ $\quad k + 3 \stackrel{?}{=} k+3$

Yes! True

---

4. Using 4¢ and 11¢ stamps.. we can make n¢ postage.

(4)	(4,4)	(11)	(4,4,4)	(4,11)	(4x4)	(2x4, 11)	(5x4)	(2x11)	(3x4, 11)	(6x4)	(4, 2x11)	(4x4, 11)	(7x4)
4	8	11	12	15	16	19	20	22	23	24	26	27	28

(2x4, 2x11)	(5x4, 11)	(3x4, 2x11)	(6x4, 11)	(3x4, 2x11)	(6x4, 11)	(4x4, 2x11)	(7x4, 11)	(4x4, 2x11)	(7x4, 11)	...
30	31	32	33	34	35	36	37	38	39	

Basis: $P(n=30)$ is true. $2\times 4 + 2\times 11$. $\quad P(n=31)$ is true. $5\times 4 + 11$.
$P(n=32)$ is true. $3\times 4 + 2\times 11$. $\quad P(n=33)$ is true. $6\times 4 + 11$.

Ind. Hyp. Assume $P(n=j)$ is true for $30 \le j \le k$.

Ind. Step Prove $P(n=30) \wedge P(n=31) \wedge \cdots \wedge P(n=k) \longrightarrow P(n=k+1)$.

Want to prove $P(n=k+1)$. Postage of $k+1$ cents can be made by adding one 4¢ stamp to the stamps needed for $k-3$ postage, if this is possible. It is, because by the Ind. Hyp. $P(n=k-3)$ is true.

This is Strong Induction!

5. $P(n)$ $n! < n^n$

Determine minimum value of n.

n	$n!$	n^n
0	1	1
1	1	1
2	2	4
3	6	27

$n > 1$

Basis $P(n=2)$ $2! \overset{?}{<} 2^2$

$2 \overset{?}{<} 4$ Yes. True

Ind. Hyp. $P(n=k)$ assume True. $k! < k^k$

Ind. Step Prove $P(n=k) \rightarrow P(n=k+1)$

$(k+1)! < (k+1)^{k+1}$: $P(n=k+1)$

Keep this in mind

by Ind. Hyp. : $k! < k^k$

multiply by $(k+1)$ $(k+1)\,k! < (k+1)\,k^k$

$(k+1)! < (k+1)\,k^k$

Need to make right hand side be

We know $(k+1)\,k^k < (k+1)(k+1)^k$ $(k < k+1)$

$= (k+1)^{k+1}$

So $(k+1)! < (k+1)^{k+1}$ Proved.

6. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = ?$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = f(n)$$

$$- \quad \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} = \frac{1}{2} f(n) \qquad \text{multiply by } \frac{1}{2}$$

$$\frac{1}{2} - \frac{1}{2^{n+1}} = f(n) - \frac{1}{2} f(n)$$

multiply by 2: $\quad 1 - \frac{1}{2^n} = 2f(n) - f(n)$

$$1 - \frac{1}{2^n} = (2-1) f(n)$$

$$1 - \frac{1}{2^n} = f(n) \qquad (P(n))$$

Prove it.

Basis: $P(n=1)$ $\quad \frac{1}{2^1} \stackrel{?}{=} 1 - \frac{1}{2^1}$

$\frac{1}{2} \stackrel{?}{=} \frac{1}{2}$ Yes. True.

Ind. Hyp: ($P(n=k)$ assumed true. $\quad \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} = 1 - \frac{1}{2^k}$

Ind. Step. Prove $P(n=k) \rightarrow P(n=k+1)$

$P(n=k+1)$ $\quad \underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k}} + \frac{1}{2^{k+1}} \stackrel{?}{=} 1 - \frac{1}{2^{k+1}}$

by Ind. Hyp.

$$1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \stackrel{?}{=} 1 - \frac{1}{2^{k+1}}$$

multiply by 2^k $\quad 2^k - 1 + \frac{1}{2} \stackrel{?}{=} 2^k - \frac{1}{2}$

$\frac{2^k}{2^{k+1}} = \frac{1}{2}$

$$2^k - \frac{1}{2} \stackrel{?}{=} 2^k - \frac{1}{2}$$

Yes. True

7. What is $1+2+4+8+\cdots+2^n=?$ for $n\geq 0$

$$
\begin{array}{rl}
1+2+4+8+\cdots+2^n & = f(n) \\
-\quad 2+4+8+\cdots+2^n+2^{n+1} & = 2f(n) \\
\hline
1 \qquad\qquad -2^{n+1} & = f(n)-2f(n) \\
1-2^{n+1} & = (1-2)f(n) \\
2^{n+1}-1 & = f(n)
\end{array}
$$

Prove it.

Basis $P(n=0)$ $\quad 2^0 \stackrel{?}{=} 2^1-1 \quad 1=1.$ True.

Ind. Hyp. $P(n=k)$ assume true. $\quad 1+2+\cdots+2^k = 2^{k+1}-1$

Ind. Step Prove $P(n=k)\to P(n=k+1)$

def of $P(n=k+1)$ $\quad \underbrace{1+2+4+\cdots+2^k}_{2^{k+1}-1}+2^{k+1} \stackrel{?}{=} 2^{k+1+1}-1$

by Ind. Hyp. $\quad 2^{k+1}-1+2^{k+1} \stackrel{?}{=} 2^{k+2}-1$

$$2\cdot 2^{k+1}-1 \stackrel{?}{=} 2^{k+2}-1$$

$$2^{k+2}-1 = 2^{k+2}-1$$

Yes True.

8. Power recursive method.

Basis. $P(n=0)$ Method returns 1. True.

Ind. Hyp. $P(n=k)$ Assume $power(x, k) = x^k$ ✓

Ind. Step. Prove $P(n=k) \rightarrow P(n=k+1)$

Method returns $x \cdot power(x, k+1-1)$

by Ind. Hyp it returns $x \cdot x^k$

returns x^{k+1} The correct answer!

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9. Tower of Hanoi

$P(n)$: $S_n = 2^n - 1$, the number of disk moves.

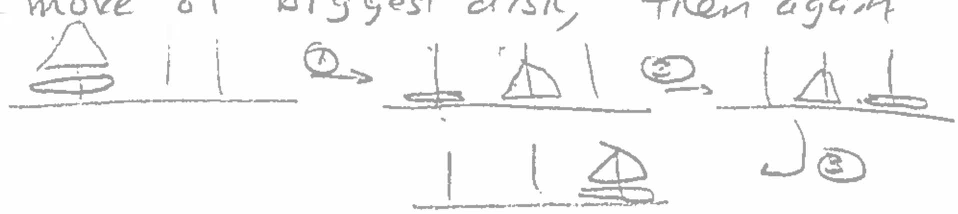
Basis $P(n=1)$: $S_1 = 2^1 - 1 = 1$ Correct. One move to solve puzzle with one disk.

Ind. Hyp. $P(n=k)$ assume true.

$$S_k = 2^k - 1$$

Ind. Step. $P(n=k) \rightarrow P(n=k+1)$

Puzzle with k+1 disks solved by moving first k disks, then one move of biggest disk, then again move k disks.



$$S_{k+1} = S_k + 1 + S_k$$
$$= 2^k - 1 + 1 + 2^k - 1 \quad \text{by Ind. Hyp.}$$
$$= 2 \cdot 2^k - 1$$
$$= 2^{k+1} - 1 \quad \text{True}$$