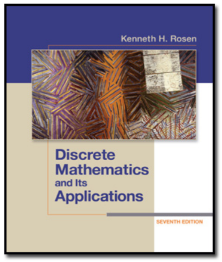
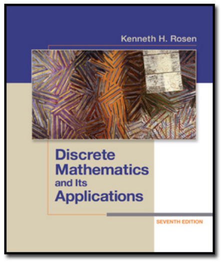


Discrete Math Review

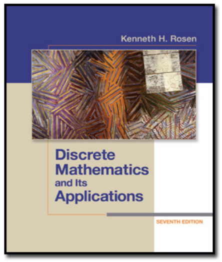
TOPICS

- Propositional and Predicate Logic
- Logical Operators and Truth Tables
- Logical Equivalences and Inference Rules



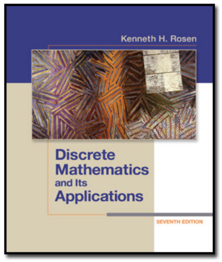
Discrete Math Review

- What you should know about discrete math before the midterm.
- Less theory, more problem solving, focuses on exam problems, use as study sheet!



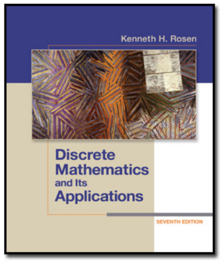
Propositional Logic

- A *proposition* is a statement that is either true or false
- Examples:
  - Fort Collins is in Nebraska (false)
  - Java is case sensitive (true)
  - We are not alone in the universe (?)
- Every proposition is true or false, but its *truth value* may be unknown



Logical Operators

- ¬ logical not (negation)
- ∨ logical or (disjunction)
- ∧ logical and (conjunction)
- ⊕ logical exclusive or
- → logical implication (conditional)
- ↔ logical bi-implication (biconditional)

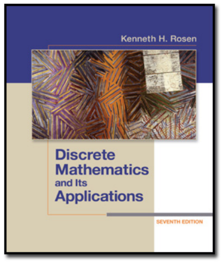


Truth Tables

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

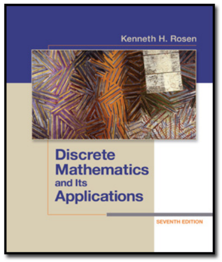
- (1) You should be able to write out the truth table for all logical operators, from memory.



Compound Propositions

- Propositions and operators can be combined into compound propositions.
- (2) You should be able to make a truth table for any compound proposition:

p	q	$\neg p$	$p \rightarrow q$	$\neg p \wedge (p \rightarrow q)$
T	T	F	T	F
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T



English to Propositional Logic

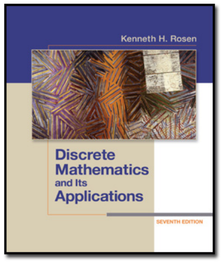
- (3) You should be able to translate natural language to logic (can be ambiguous!):
- English:

  "If the car is out of gas, then it will stop"
- Logic:

  p equals "the car is out of gas"

  q equals "the car will stop"

  $p \rightarrow q$



Propositional Logic to English

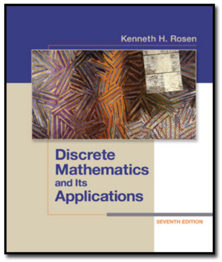
- (4) You should be able to translate propositional logic to natural language:
- Logic:

  p equals “it is raining”

  q equals “the grass will be wet”

  $p \rightarrow q$
- English:

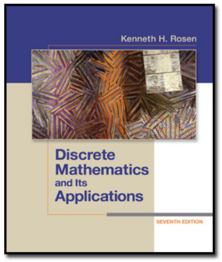
  “If it is raining, the grass will be wet.”



Logical Equivalences: Definition

- Certain propositions are equivalent (meaning they share exactly the same truth values):
- For example:

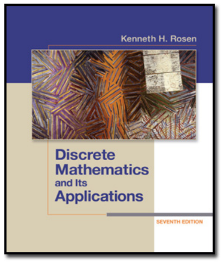
$\neg(p \land q) \equiv \neg p \lor \neg q$	De Morgan's
$(p \land T) \equiv p$	Identity Law
$(p \land \neg p) \equiv F$	Negation Law



Logical Equivalences: Truth Tables

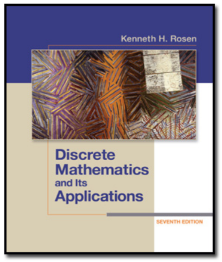
- (5) And you should know how to prove logical equivalence with a truth table
- For example: $\neg(p \land q) \equiv \neg p \lor \neg q$

p	q	$\neg p$	$\neg q$	$(p \land q)$	$\neg(p \land q)$	$\neg p \lor \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T



Logical Equivalences: Review

- (6) You should understand the logical equivalences and laws on the course web site.
- You should be able to prove any of them using a truth table that compares the truth values of both sides of the equivalence.
- Memorization of the logical equivalences is not required in this class.



Logical Equivalences (Rosen)

Logical Equivalences

Idempotent Laws

$p \lor p \equiv p$

$p \land p \equiv p$

DeMorgan's Laws

$\neg(p \land q) \equiv \neg p \lor \neg q$

$\neg(p \lor q) \equiv \neg p \land \neg q$

Distributive Laws

$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$

$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$

Double Negation

$\neg(\neg p) \equiv p$

Absorption Laws

$p \lor (p \land q) \equiv p$

$p \land (p \lor q) \equiv p$

Associative Laws

$(p \lor q) \lor r \equiv p \lor (q \lor r)$

$(p \land q) \land r \equiv p \land (q \land r)$

Commutative Laws

$p \lor q \equiv q \lor p$

$p \land q \equiv q \land p$

Implication Laws

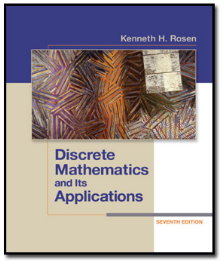
$p \rightarrow q \equiv \neg p \lor q$

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

Biconditional Laws

$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$

$p \leftrightarrow q \equiv \neg q \leftrightarrow \neg p$



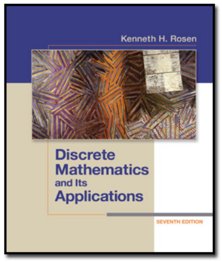
Transformation via Logical Equivalences

(7) You should be able to transform propositions using logical equivalences.

Prove: $\neg p \lor (p \land q) \equiv \neg(p \land \neg q)$

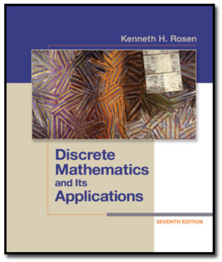
$$\begin{aligned} \neg p \lor (p \land q) &\equiv (\neg p \lor p) \land (\neg p \lor q) \\ &\equiv T \land (\neg p \lor q) \\ &\equiv (\neg p \lor q) \\ &\equiv \neg(p \land \neg q) \end{aligned}$$

- Distributive law
- Negation law
- Domination law
- De Morgan's Law



Vocabulary

- (8) You should memorize the following vocabulary:
  - A *tautology* is a compound proposition that is always true.
  - A *contradiction* is a compound proposition that is always false.
  - A *contingency* is neither a tautology nor a contradiction.
- And know how to decide the category for a compound proposition.



Examples

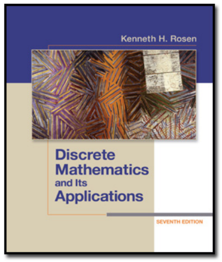
p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Result is always true, no matter what A is

Therefore, it is a ***tautology***

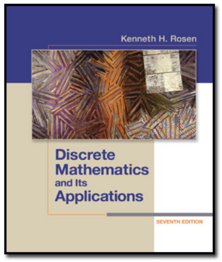
Result is always false, no matter what A is

Therefore, it is a ***contradiction***



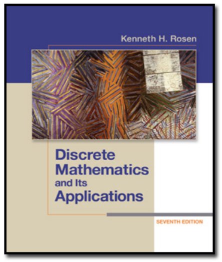
Logical Proof

- Given a set of *axioms*
  - Statements asserted to be true
- Prove a *conclusion*
  - Another propositional statement
- In other words:
  - Show that the conclusion is true …
  - … whenever the axioms are true



Logical Proof

- (9) You should be able to perform a logical proof via truth tables.
- (10) You should be able to perform a logical proof via inference rules.
- Both methods are described in the following slides.



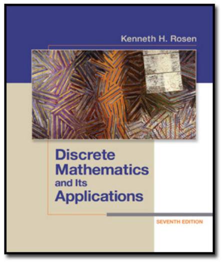
Method 1: Proof by Truth Table

- Prove that $p \rightarrow q$, given $\neg p$

p	q	$\neg p$	$p \rightarrow q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

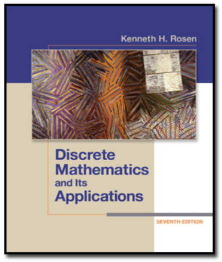
For all rows in which axiom is true, conclusion is true

Thus the conclusion follows from axiom



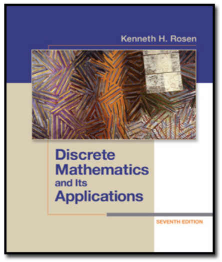
Method 2: Proof using Rules of Inference

- A *rule of inference* is a proven relation: when the left hand side (LHS) is true, the right hand side (RHS) is also true.
- Therefore, if we can match an axiom to the LHS by substituting propositions, we can assert the (substituted) RHS



Applying rules of inference

- Example rule: p, $p \rightarrow q$ $\therefore$ q
  - Read as "p and $p \rightarrow q$, therefore q"
  - This rule has a name: *modus ponens*
- If you have axioms r, $r \rightarrow s$
  - Substitute r for p, s for q
  - Apply modus ponens
  - Conclude s



Modus Ponens

- If p, and p implies q, then q

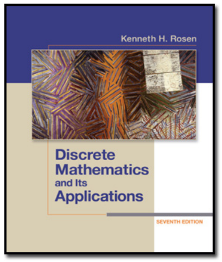
Example:

p = it rains, q = the lawn will be wet

$p \rightarrow q$, when it rains, then the lawn will be wet

"Given the above, if it rains, the lawn must be wet"

Even if we cannot observe the lawn!



Modus Tollens

- If not q and p implies q, then not p

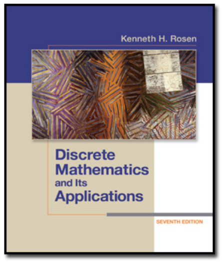
Example:

p = it rains, q = the lawn will be wet

$p \rightarrow q$, when it rains, then the lawn will be wet

"Given the above, if the lawn is not wet, it cannot have rained."

Even if we did not observe the weather!



Rules of Inference (Rosen)

Rules of Inference

Modus Ponens

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline q \end{array}$$

Modus Tollens

$$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \neg p \end{array}$$

Hypothetical Syllogism

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline p \rightarrow r \end{array}$$

Addition

$$\begin{array}{l} p \\ \hline p \vee q \end{array}$$

Resolution

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline q \vee r \end{array}$$

Disjunctive Syllogism

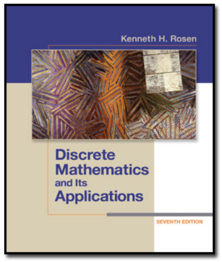
$$\begin{array}{l} p \vee q \\ \neg p \\ \hline q \end{array}$$

Simplification

$$\begin{array}{l} p \wedge q \\ \hline p \end{array}$$

Conjunction

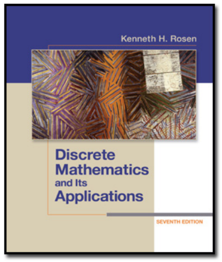
$$\begin{array}{l} p \\ q \\ \hline p \wedge q \end{array}$$



A Simple Proof: Problem Statement

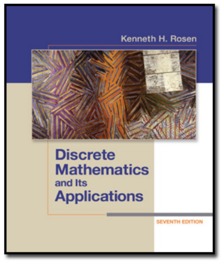
Example of a complete proof using inference rules, from English to propositional logic and back:

- If you don’t go to the store, then you cannot not cook dinner. (axiom)
- If you cannot cook dinner or go out, you will be hungry tonight. (axiom)
- You are not hungry tonight, and you didn’t go to the store. (axiom)
- You must have gone out to dinner. (conclusion)



A Simple Proof: Logic Translation

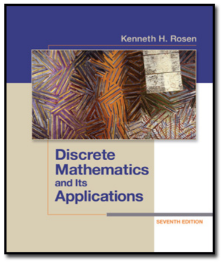
- p: you go to the store
- q: you can cook dinner
- r: you will go out
- s: you will be hungry
- AXIOMS: $\neg p \rightarrow \neg q$, $\neg(q \vee r) \rightarrow s$, $\neg s$, $\neg p$
- CONCLUSION: r



A Simple Proof: Applying Inference

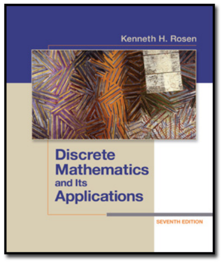
1.	$\neg p \rightarrow \neg q$	Axiom
2.	$\neg(q \vee r) \rightarrow s$	Axiom
3.	$\neg s$	Axiom
4.	$\neg p$	Axiom
5.	$\neg q$	Modus Ponens (1, 4)
6.	$q \vee r$	Modus Tollens (2, 3)
7.	r	Disjunctive Syllogism (5, 6)

CONCLUSION: You must have gone out to dinner!



Predicate Logic

- (11) You should recognize predicate logic symbols, i.e. quantifications.
- Quantification express the extent to which a predicate is true over a set of elements:
  - Universal ∀, “for all”
  - Existential ∃, “there exists”
- (12) You should able to translate between predicate logic and English, in both directions.



Predicate Logic (cont'd)

- Specifies a proposition (and optionally a domain), for example:
  - $\exists x \in N, -10 < x < -5$ // False, since no negative x
  - $\forall x \in N, x > -1$ // True, since no negative x
- (13) Must be able to find examples
  - to prove $\exists$, e.g. $\exists x \in Z, -1 < x < 1$, $x = 0$
- (14) Must be able to find counterexamples
  - to disprove $\forall$, e.g. $\forall x \in Z, x > -1$, $x = -2$
  - to prove $\forall$, you need to make a mathematical argument that holds for all values of x.